Laplace Operator and Heat Kernel for Shape Analysis

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Laplace Operator

- on $\mathbb{R}^k$, the standard Laplace operator:
  - $\Delta_{\mathbb{R}^k} f := \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_k^2}$
  - $\Delta_{\mathbb{R}^k} f := \textbf{div} \nabla f$

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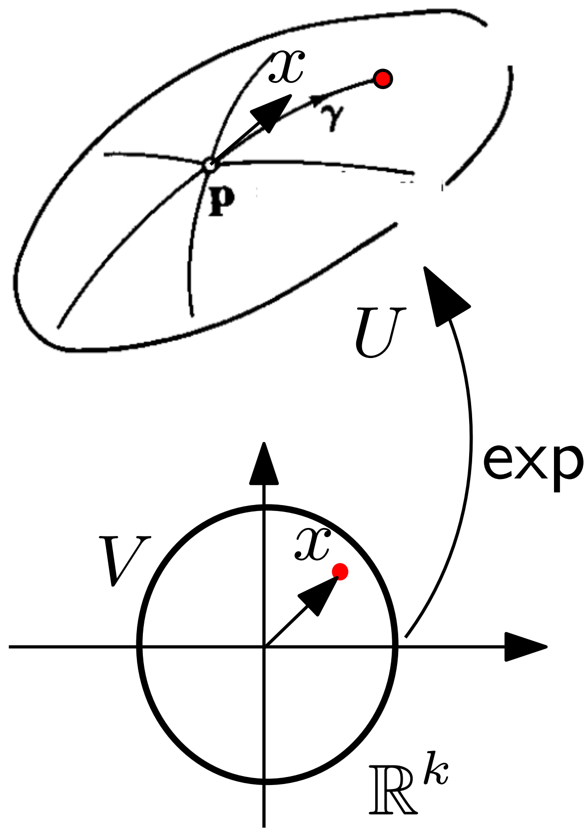
  - $\Delta_{\mathbb{R}^k} f := \textbf{div} \nabla f$
- on a Riemannian manifold (M, g), Laplace-Beltrami operator:
  - $\Delta_M f := \textbf{div} \nabla f$
  - $\Delta_M f := \frac{1}{\sqrt{\det g}} \sum_{i,j} \frac{\partial}{\partial x^j} (g^{ij} \sqrt{\det g} \frac{\partial f}{\partial x^i})$

Laplace Operator

- on a Riemannian manifold, Laplace-Beltrami operator:
  - Exponential map: $\exp : V \subset \mathbb{R}^k \to U$ by $\exp(x) = \gamma(p, x, 1)$
  - $\tilde{f}(x) = f(\exp(x))$
  - $\Delta_M f := \Delta_{\mathbb{R}^k} \tilde{f} = \frac{\partial^2 \tilde{f}}{\partial x_1^2} + \cdots + \frac{\partial^2 \tilde{f}}{\partial x_k^2}$
  - Laplace-Beltrami operator is invariant under the map preserving geodesics



Eigenvalues and eigenfunctions

- $\Delta_M \phi = \lambda \phi$
  - For compact manifold, Δ_M is compact
  - $0 \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$, ∞ is the only accumulating point

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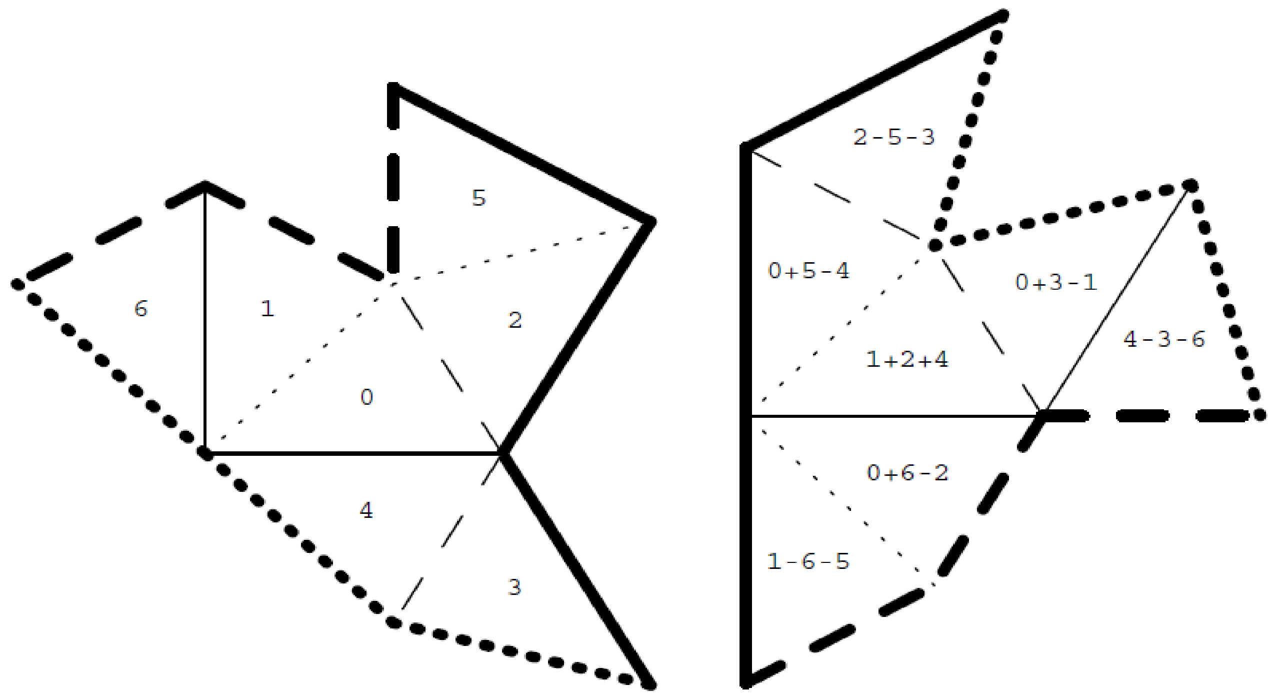
- Spectrum: eigenvalues
  - $\lambda_n \sim 4\pi^2 (\frac{n}{w_d Vol(M)})^{2/d}$ as $n \uparrow \infty$
  - heat trace: $\sum_i e^{\lambda_i t} = \frac{1}{(4\pi t)^{d/2}} \sum_i c_i t$.
    - $c_0 = vol(M), c_1 = \frac{1}{3} \int s$.

Spectrum

- isospectrality
  - “Can you hear the shape of a drum” [Kac 1966]
  - “Does the spectrum determines the shape upto isometry

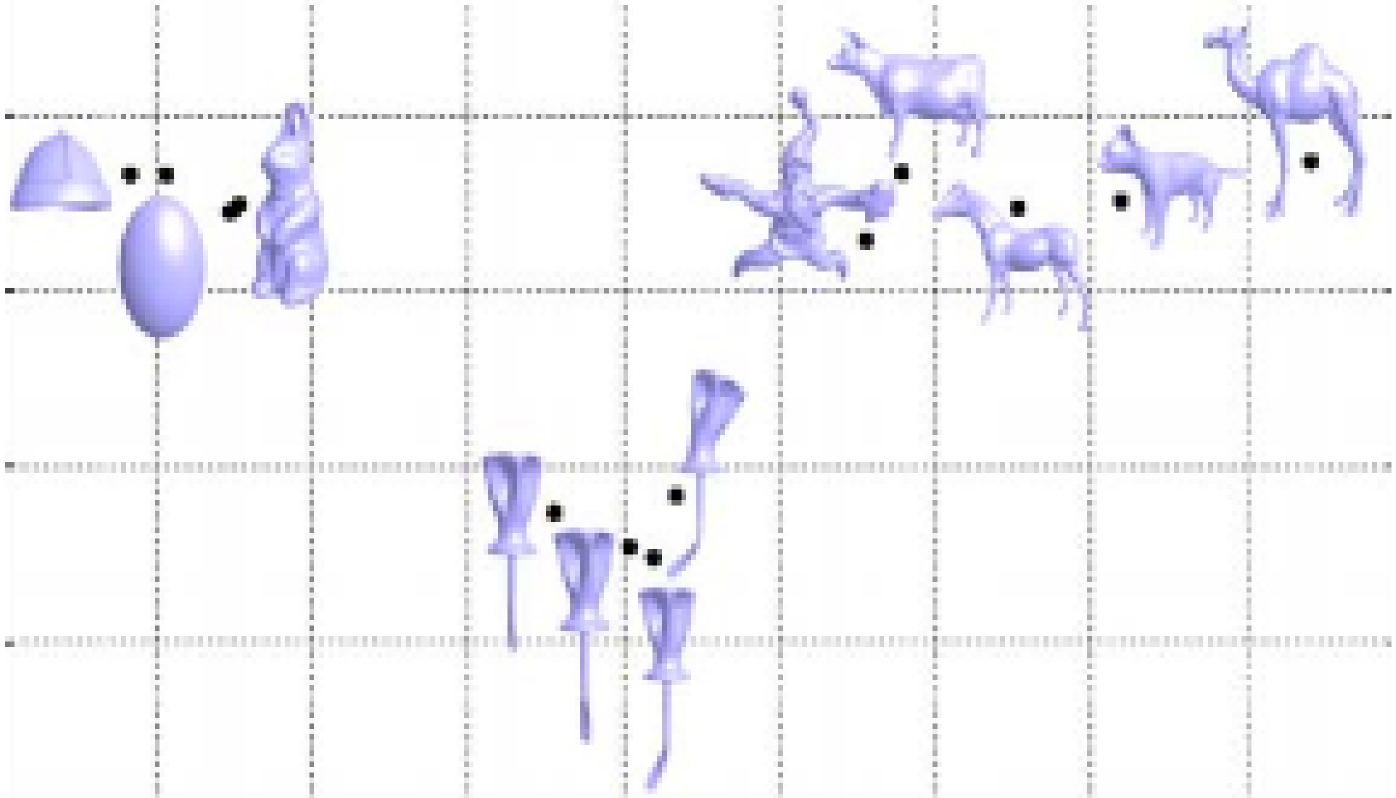
Spectrum

- isospectrality
  - "Can you hear the shape of a drum" [Kac 1966]
  - "Does the spectrum determines the shape upto isometry
  - negative [Gordon et al. 1992, Buser et al. 1992]



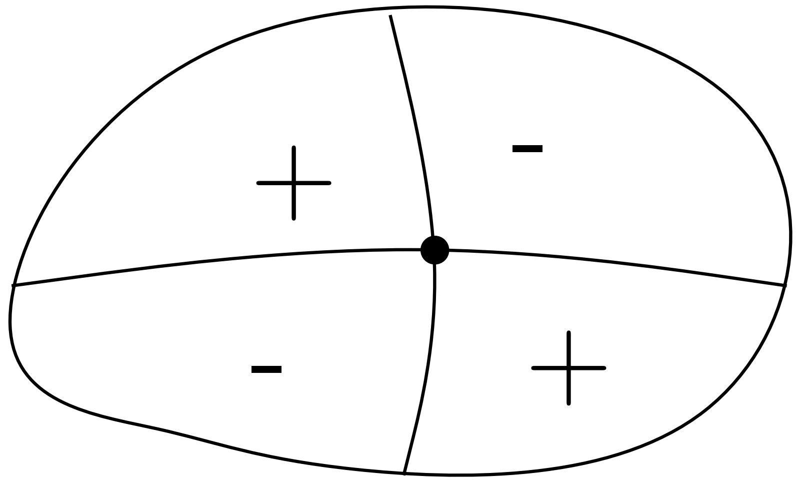
Spectrum

- Spectrum: shape DNA [Reuter et al. 2006]



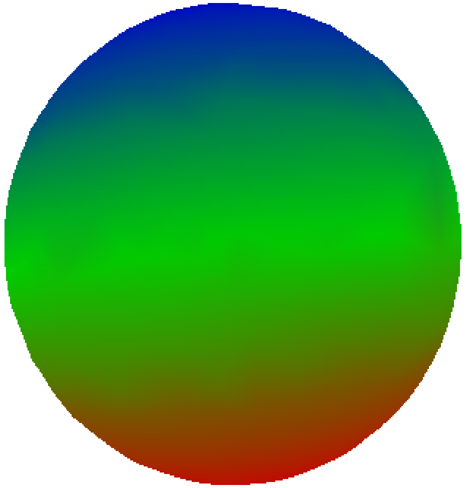
Eigenfunctions

- $\Delta_M \phi = \lambda \phi$, (M, g) is C^∞-manifold
  - nodal set: $\phi^{-1}(0)$,
    nodal domain: the connected component of $M \setminus \phi^{-1}(0)$
  - Nodal domain theorem [Courant and Hilbert 1953, Cheng 1976]: $\#$ of nodal domains of the i-th eigenfunction $\leq i + 1$
  - Properties of Nodal Set [Cheng 1976]: Except on a closed set of lower dimension(i.e., dim $< d - 1$) the nodal set off forms an (d - 1)-dim C^∞-manifold.

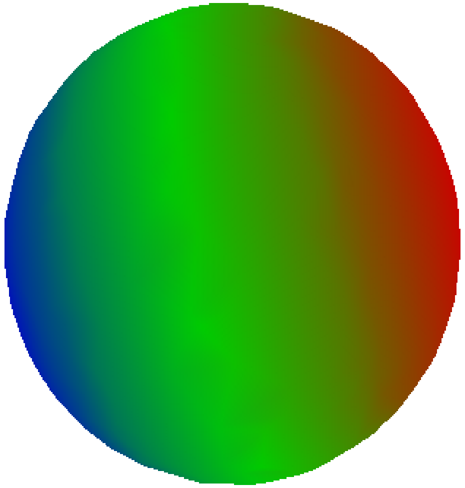


Examples

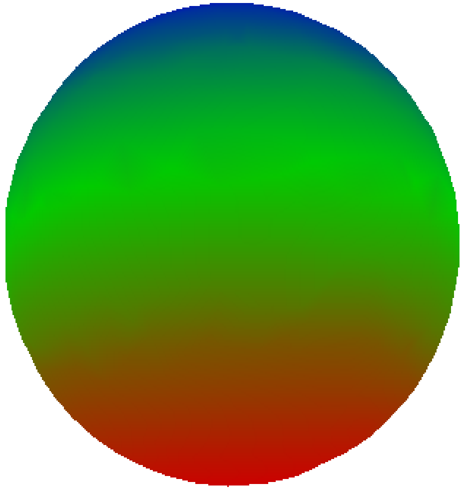
- sphere $(x^2 + y^2 + z^2 = 1)$



$\lambda_1 = 2(1.91)$



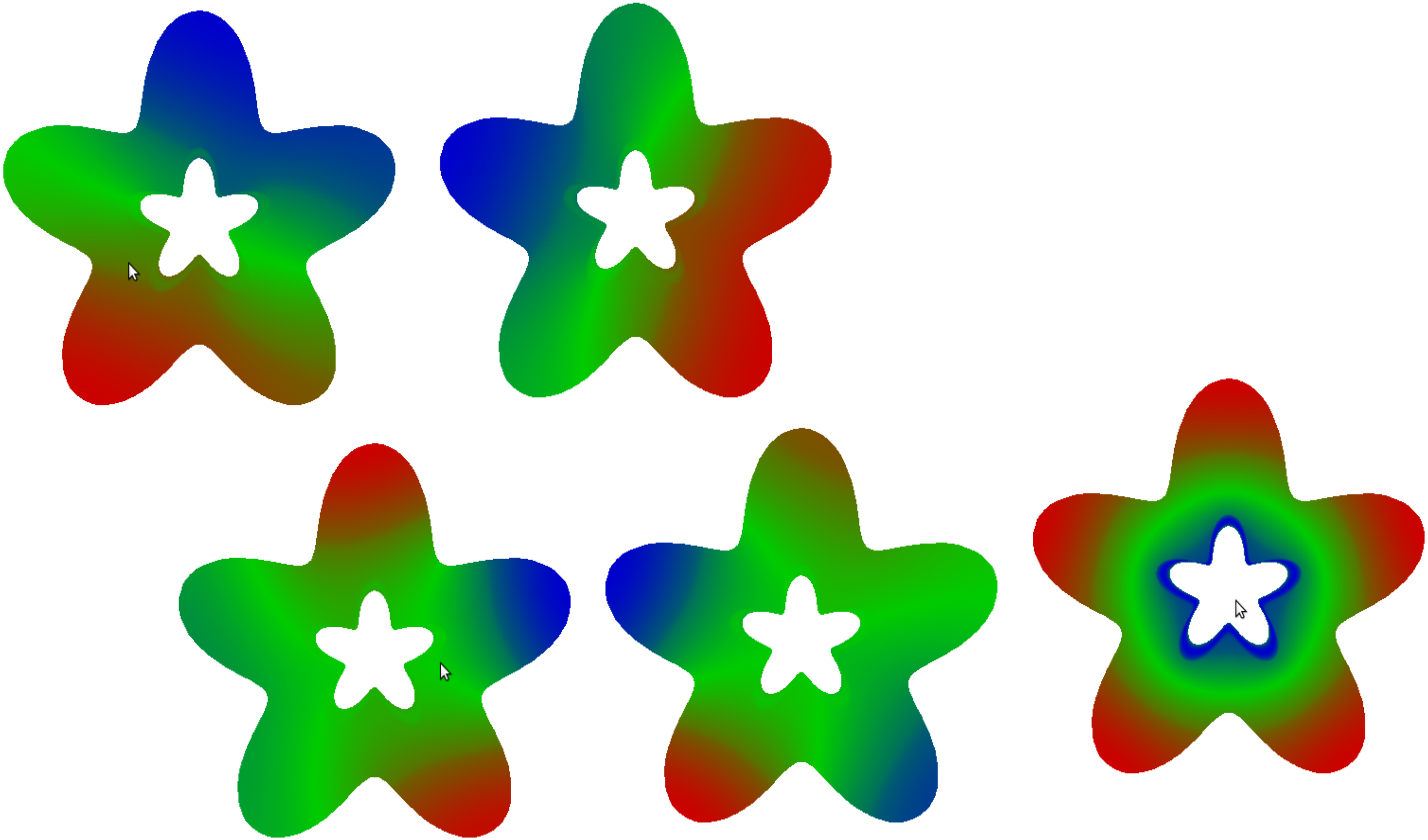
$\lambda_2 = 2(1.92)$



$\lambda_1 = 2(1.93)$

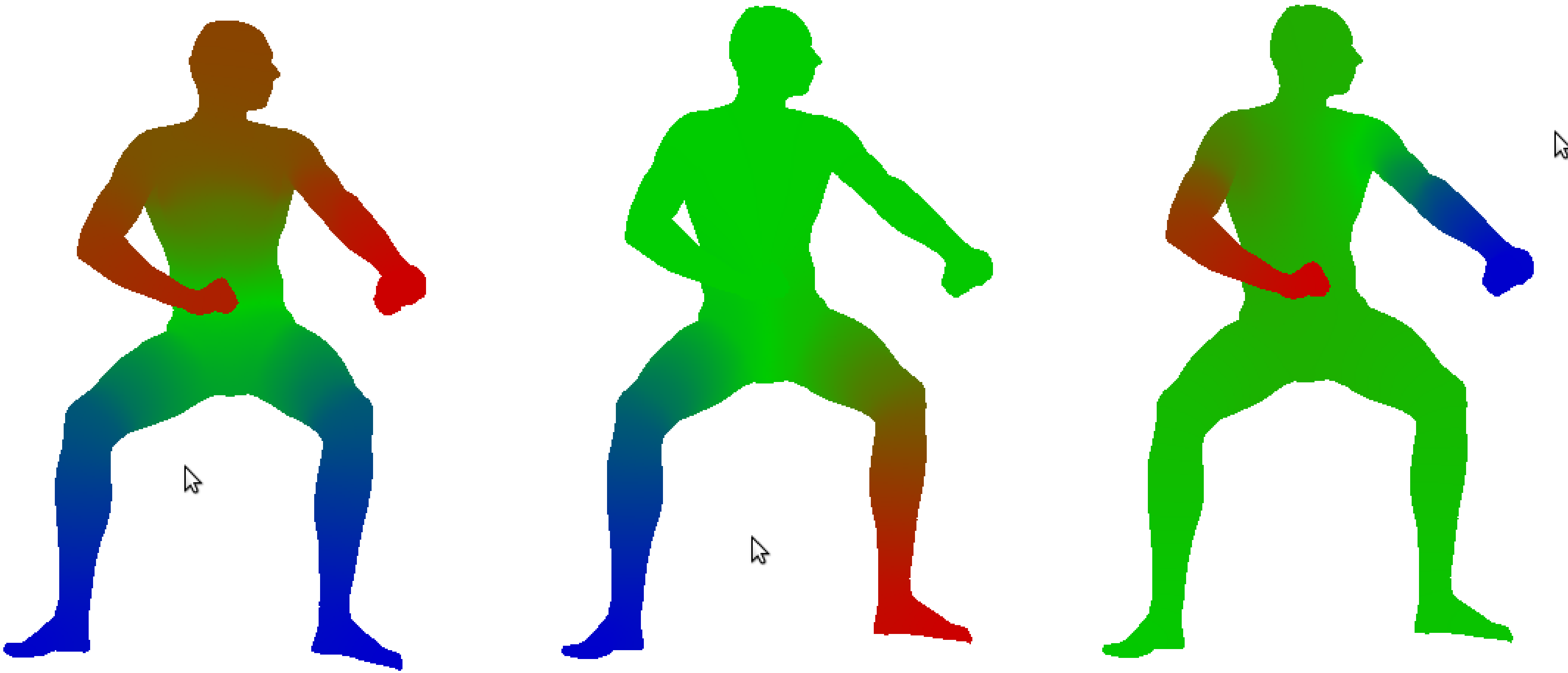
Examples

- star



Examples

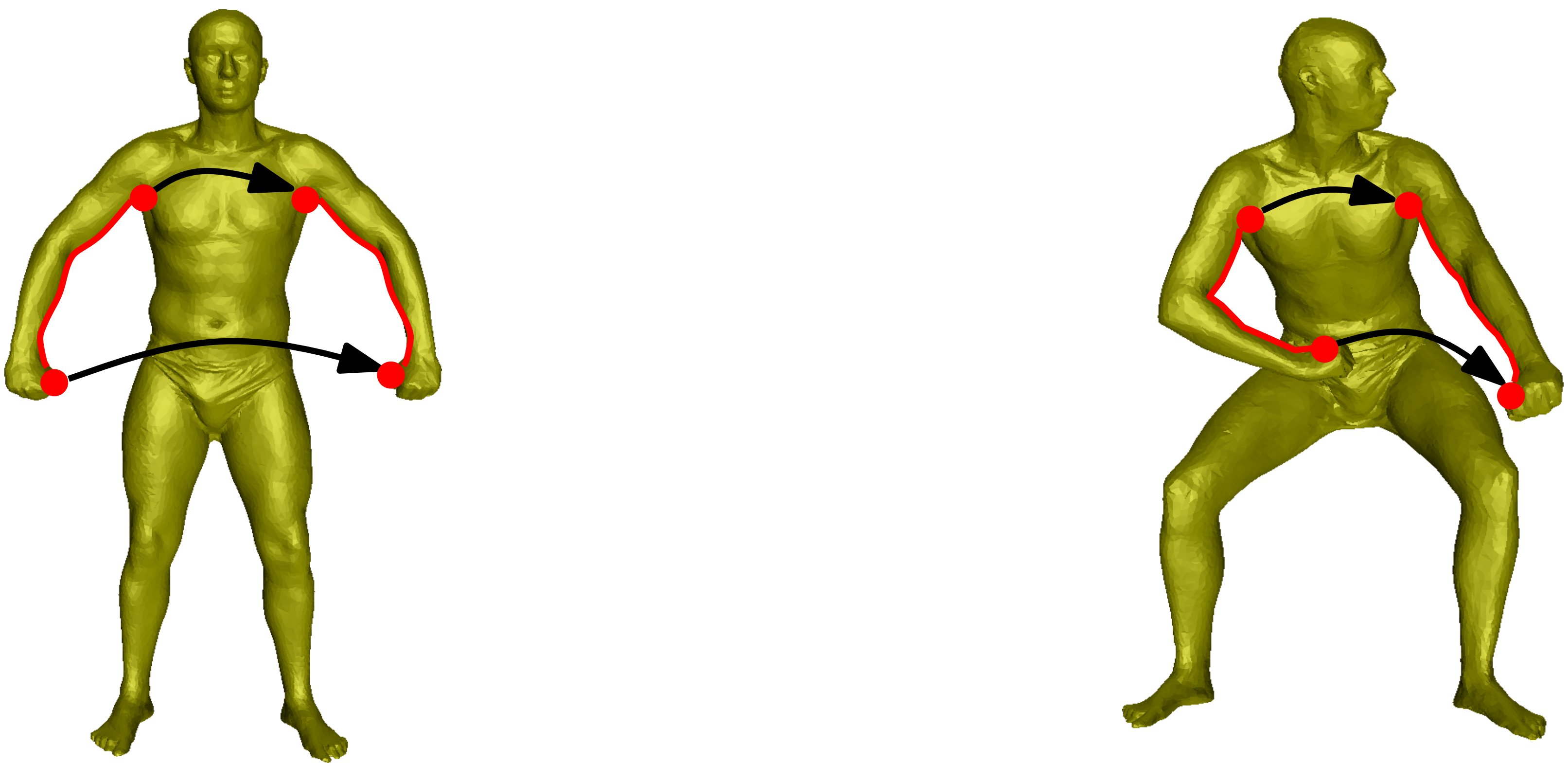
- human



Intrinsic Symmetry Detection

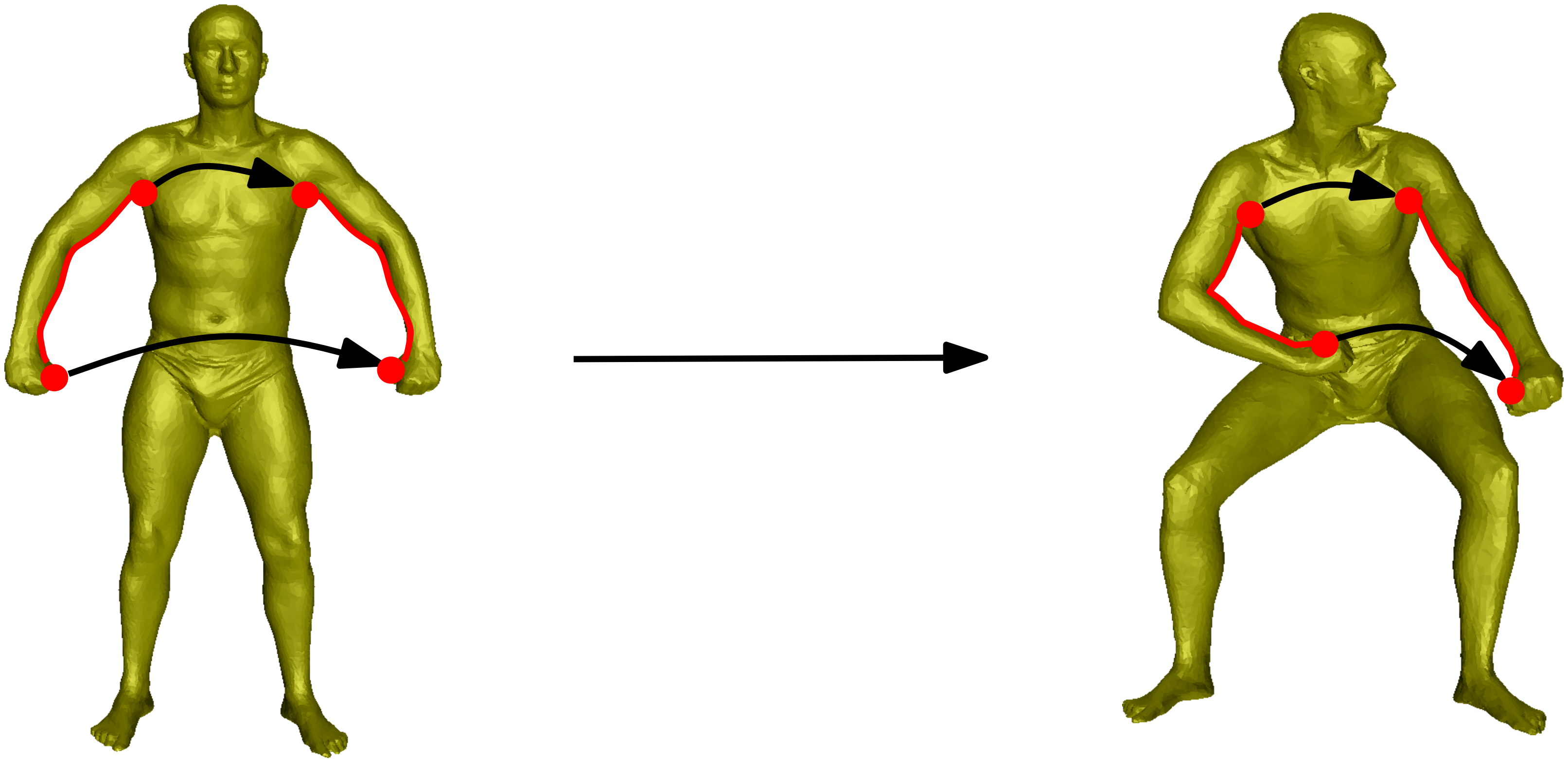
Intrinsic Symmetry

- intrinsic symmetry: a self map preserving geodesic distances



Intrinsic Symmetry

- intrinsic symmetry: a self map preserving geodesic distances
  - invariant under non-rigid transformations



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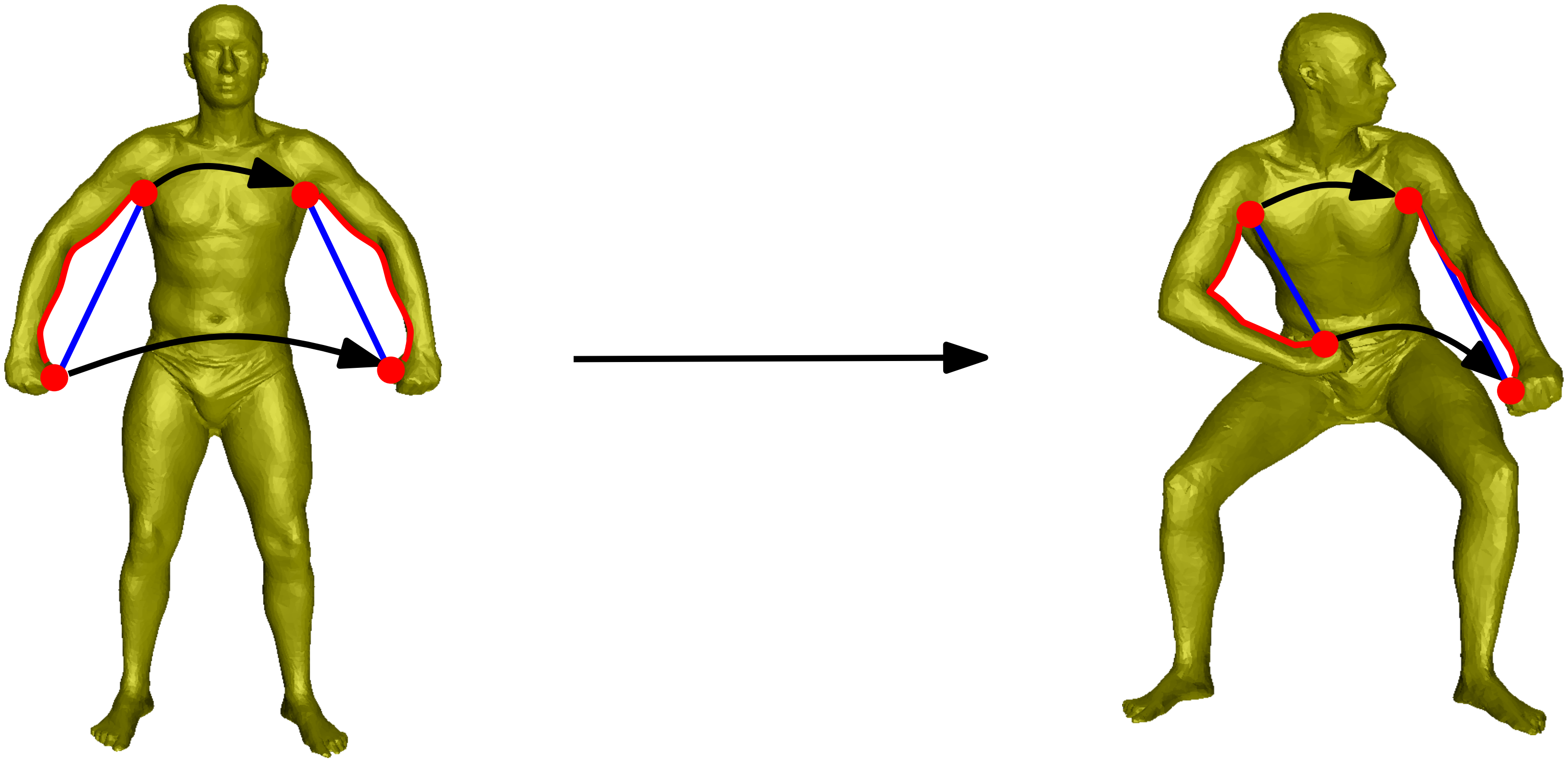
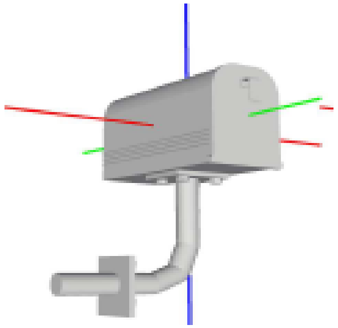


- extrinsic symmetry: rotation and reflection
  - preserve Euclidean distances
  - invariant only under rigid transformations

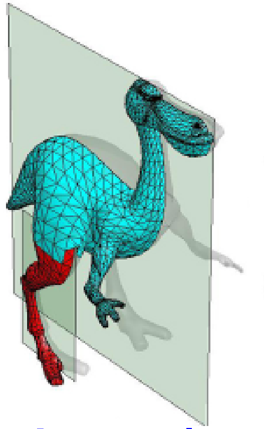
Related Work

- extrinsic symmetry

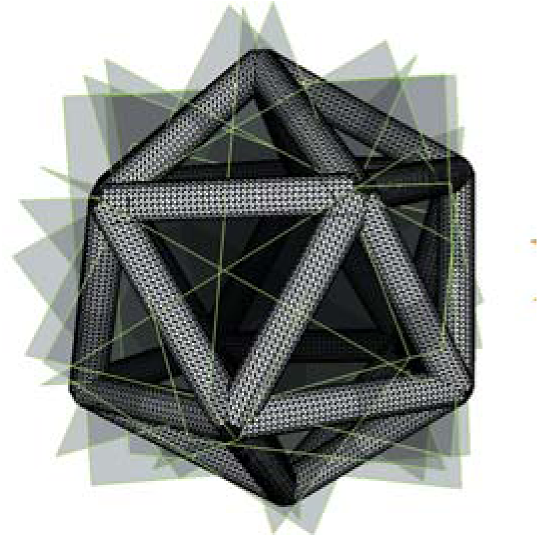


[Podolak et al. 06]

[Mitra et al. 06]



[Shimari et al. 06]



[Martinet et al. 07]

- intrinsic symmetry
  - difficulty: no simple characterization

Global Point Signature

- our strategy: reduce intrinsic to extrinsic
- our tool: eigenfunctions ϕ_i and eigenvalues λ_i of Δ_M

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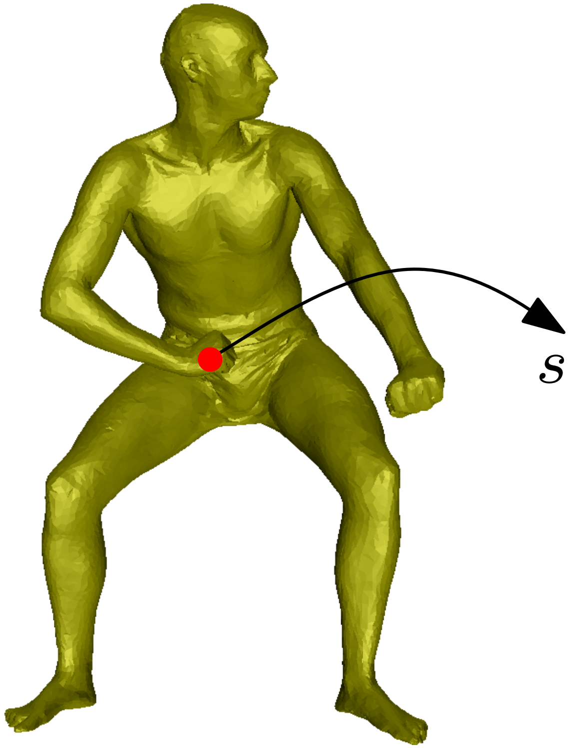
- for each point p on M, its GPS [Rustamov 07]



$$s(p) = \begin{pmatrix} \frac{\phi_1(p)}{\sqrt{\lambda_1}} \\ \frac{\phi_2(p)}{\sqrt{\lambda_2}} \\ \vdots \\ \frac{\phi_i(p)}{\sqrt{\lambda_i}} \\ \vdots \end{pmatrix}$$

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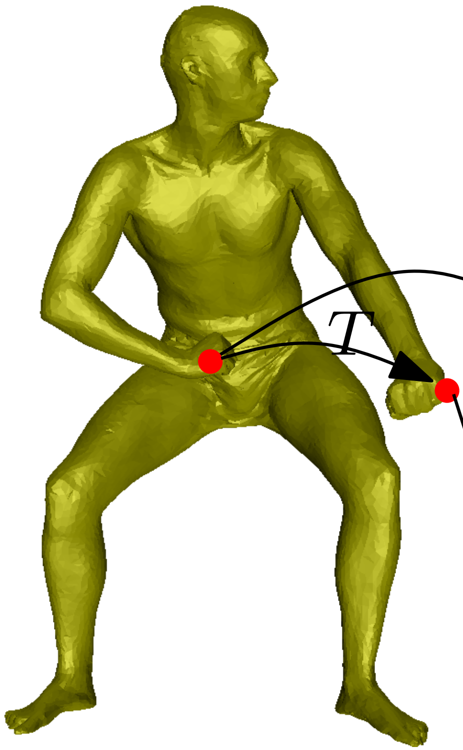

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- ○ relation between $s(p)$ and $s(T(p))$?

Transforming Theorem

Theorem: For a compact manifold M, T is an intrinsic symmetry if and only if there is a transformation R such that $R(s(p)) = s(T(p))$ for each point $p \in M$ and R restricting to any eigenspace is a rigid transformation.

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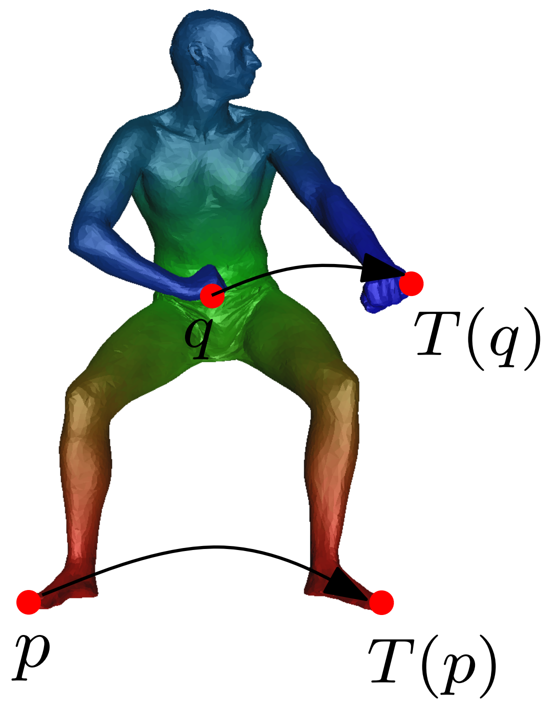
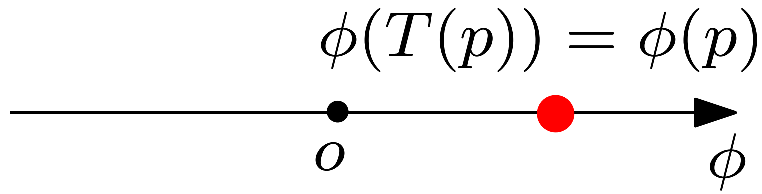
- "only if" part

Eigenfunctions and Intrinsic Symmetry

1. $\phi = \phi \circ T$: **positive** eigenfunction
2. $\phi = -\phi \circ T$: **negative** eigenfunction
3. λ is a repeated eigenvalue





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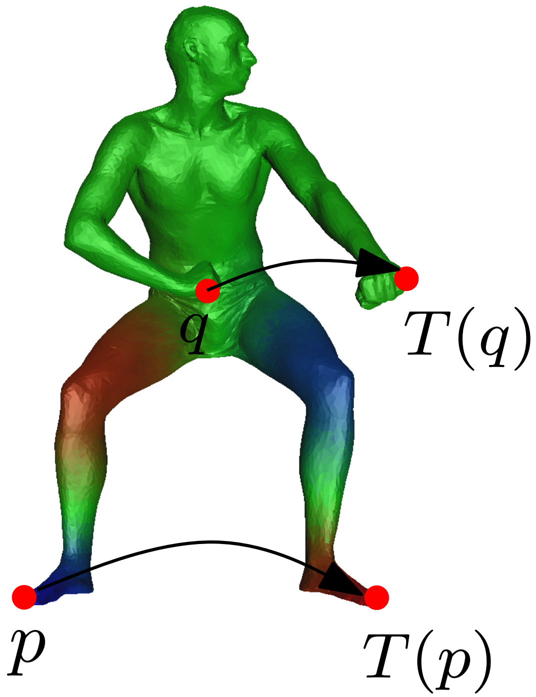

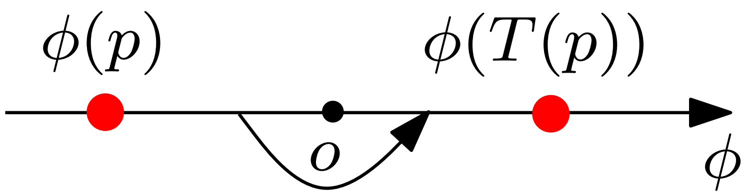

reflection

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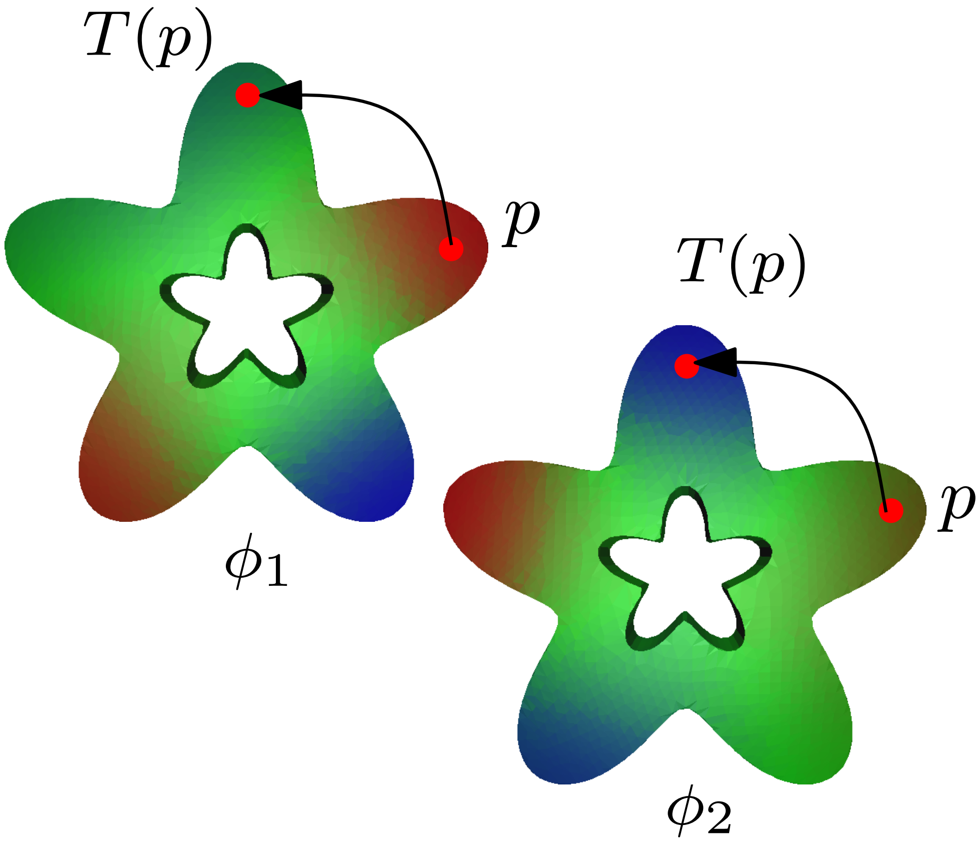

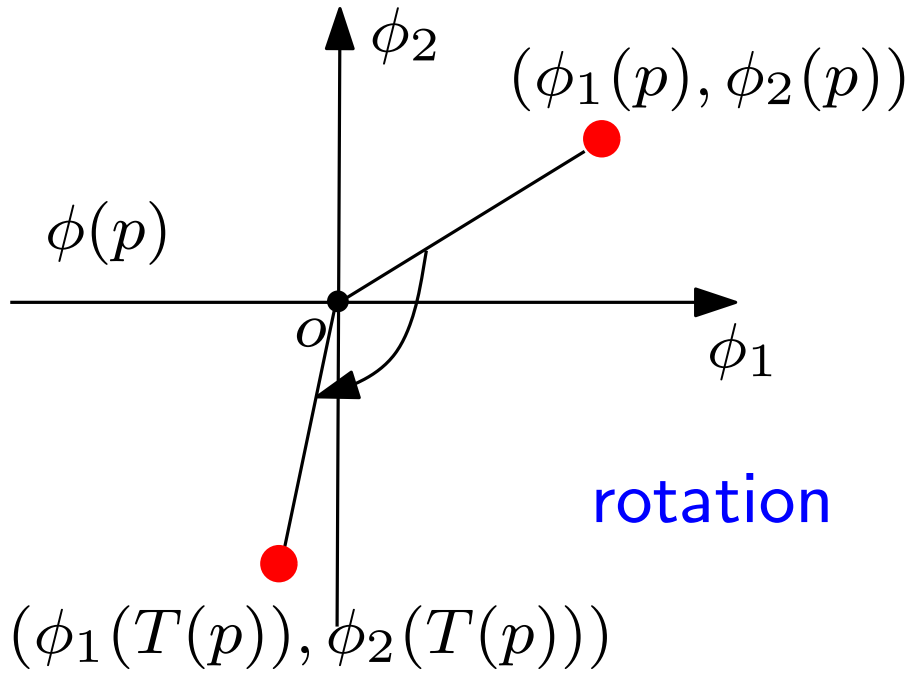

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  - T is an identity $\Leftrightarrow$ R is an identity

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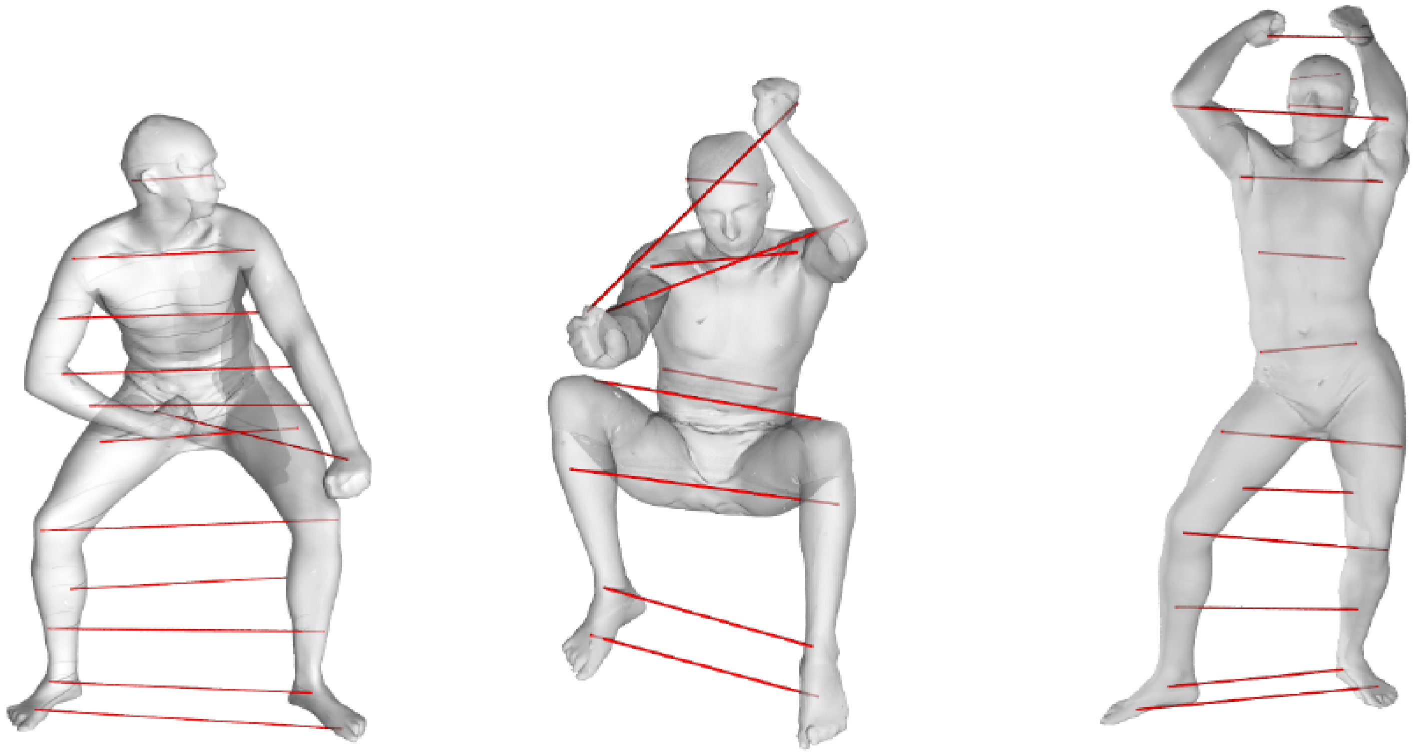
- one to one correspondence between T and R
  - T is an identity $\Leftrightarrow$ R is an identity
- detection of intrinsic symmetry reduced to that of extrinsic rigid transformation
  - detection of extrinsic symmetry
    [Rus07, PSG06, MGP06, MSHS06]

Results

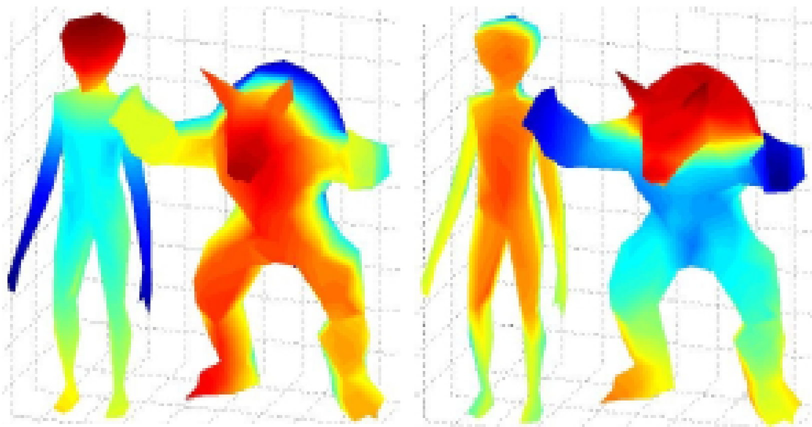


(a) (b) (c)

scans of a real person
(SCAPE dataset)

Limitation of Global Point Signature

- For any x, its GPS [Rus07]:
  $\text{GPS}_x = \left(\phi_1(x)/\sqrt{\lambda_1}, \phi_2(x)/\sqrt{\lambda_2}, \cdots, \phi_i(x)/\sqrt{\lambda_i}, \cdots\right)$
  - global
  - not unique
    - orthonormal transformation within eigenspace
    - eigenfunction switching [GVL96]



courtesy of Jain and Zhang

Heat Kernel Signature

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  - $\mathsf{HKS}_x : \mathbb{R}^+ \to \mathbb{R}^+$

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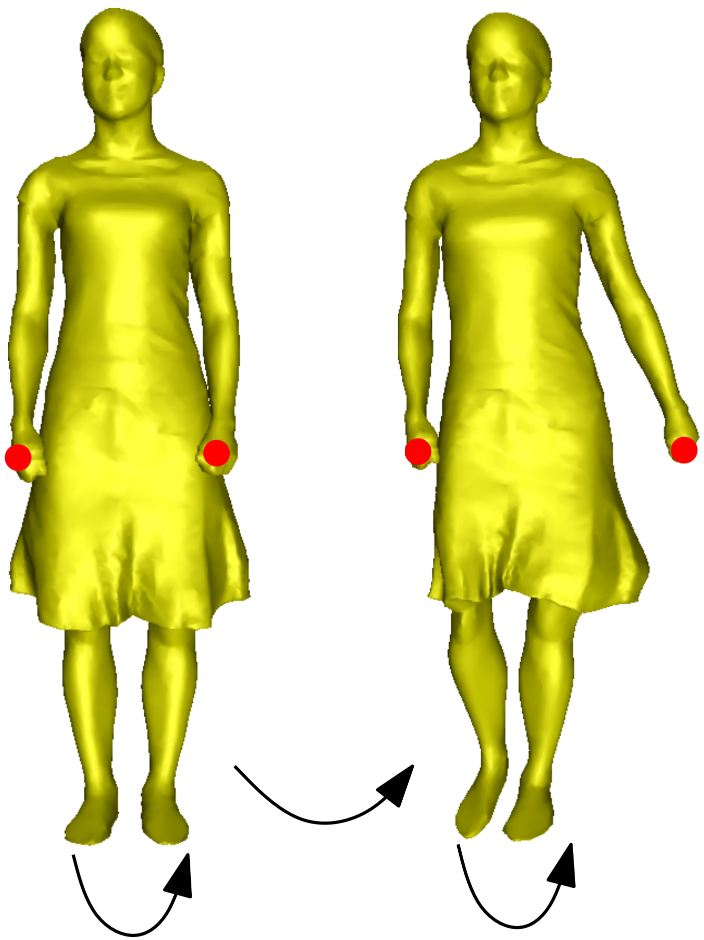

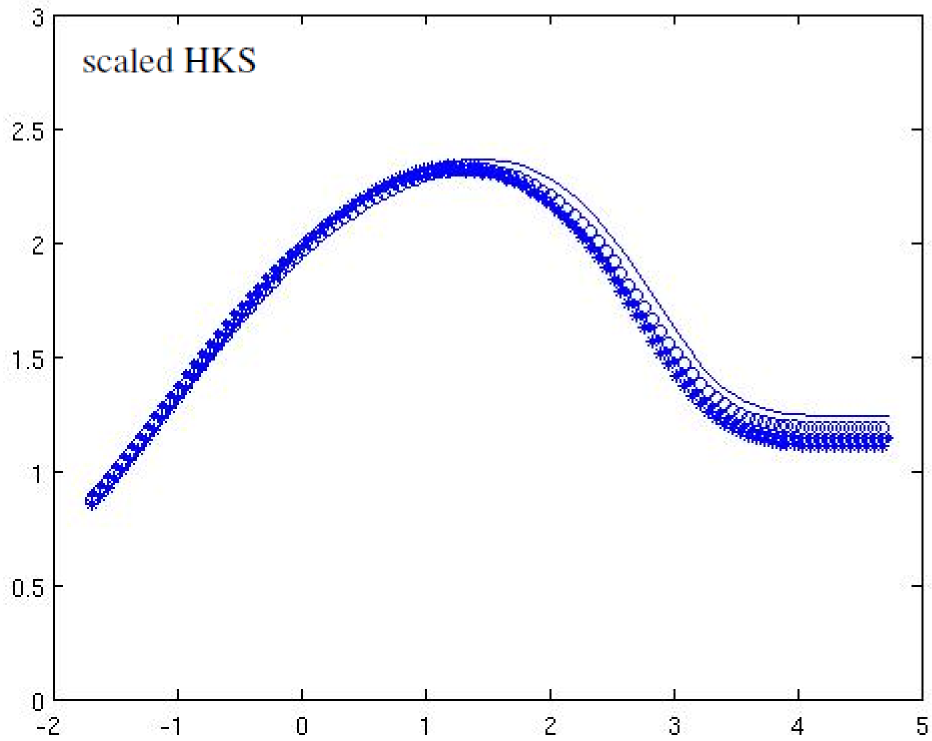

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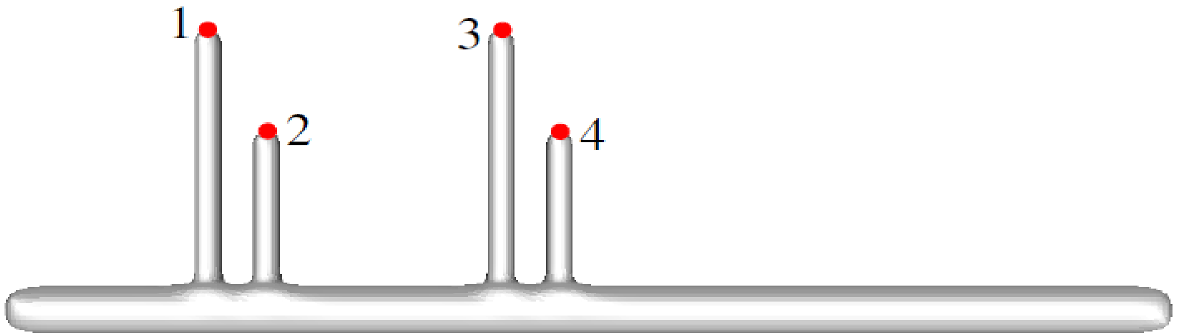
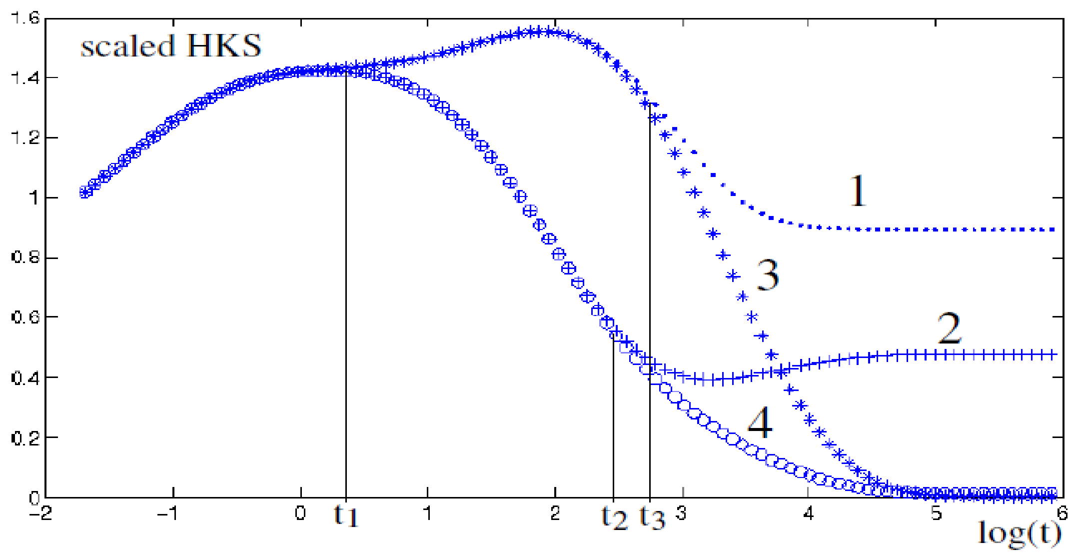
- multi-scale





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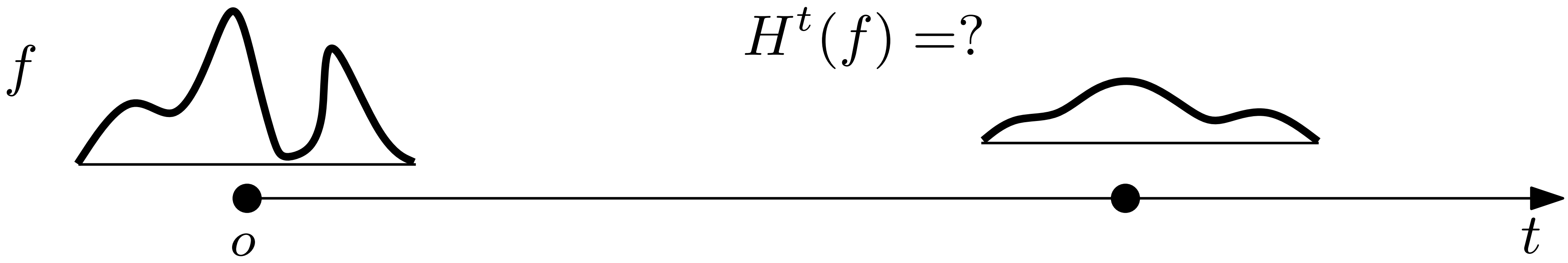
- define HKS for any point x as a function on $\mathbb{R}^+$:
  - $\text{HKS}_x : \mathbb{R}^+ \to \mathbb{R}^+$
- isometric invariant
- multi-scale
- informative
  - $\{HKS_x\}_{x \in M}$ characterizes almost all shapes up to isometry.

Heat Diffusion Process

- how heat diffuses over time



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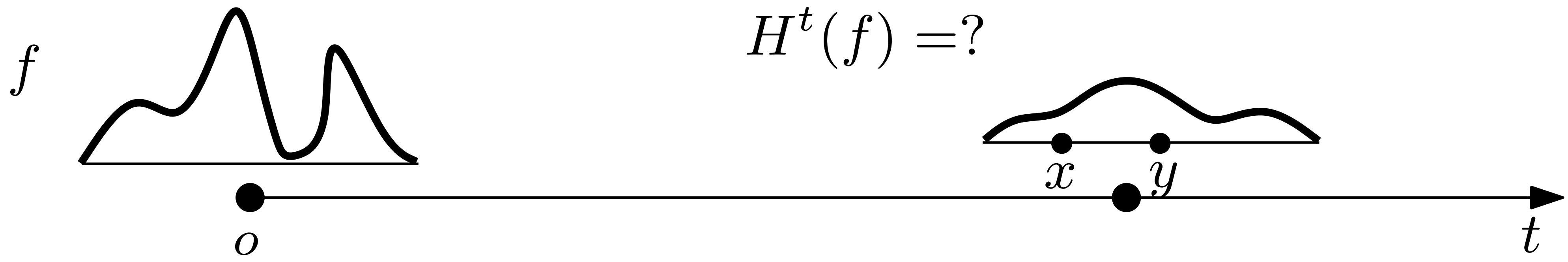

- heat kernel $k_t(x, y) : \mathbb{R}^+ \times M \times M \to \mathbb{R}^+$
  - heat transferred from y to x in time t
  - $H^t f(x) = \int_M k_t(x, y) f(y) dy$

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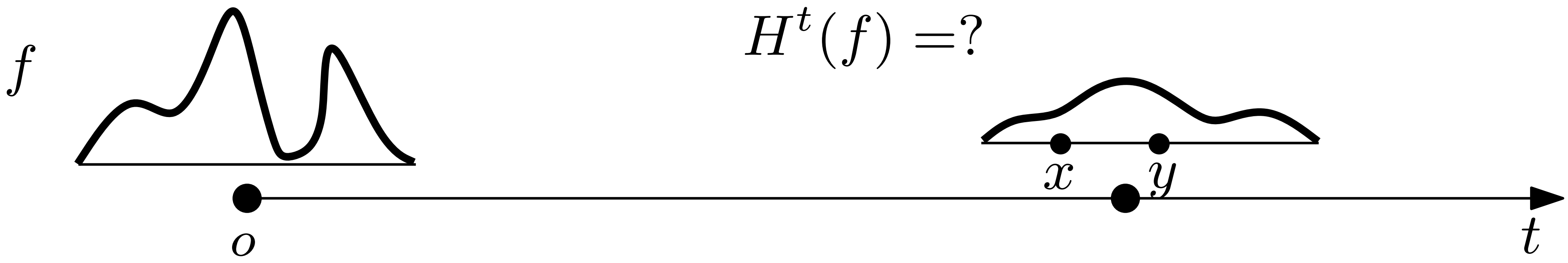

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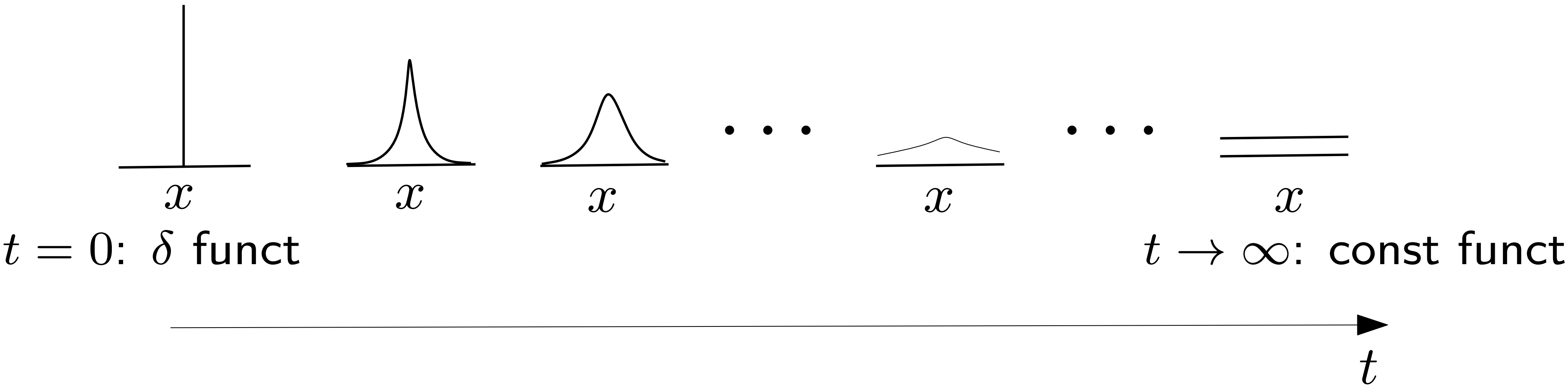
  - $k_t(x, y) = \sum_i e^{-\lambda_i t} \phi_i(x) \phi_i(y)$

Heat Kernel

- characterize shape up to isometry
  - $T : M \to N$ is isometric iff $k_t(x, y) = k_t(T(x), T(y))$.
  - heat kernel recovers geodesic distances.
    - $d_M^2(x, y) = -4 \lim_{t \to 0} t \log k_t(x, y)$
  - heat diffusion process governed by heat equation.
    - $\Delta_M u(t, x) = -\frac{\partial u(t,x)}{\partial t}$
- generate a Brownian motion on a manifold.

Heat Kernel

- multi-scale
  - for any x, a family of functions $\{k_t(x,\cdot)\}_t$



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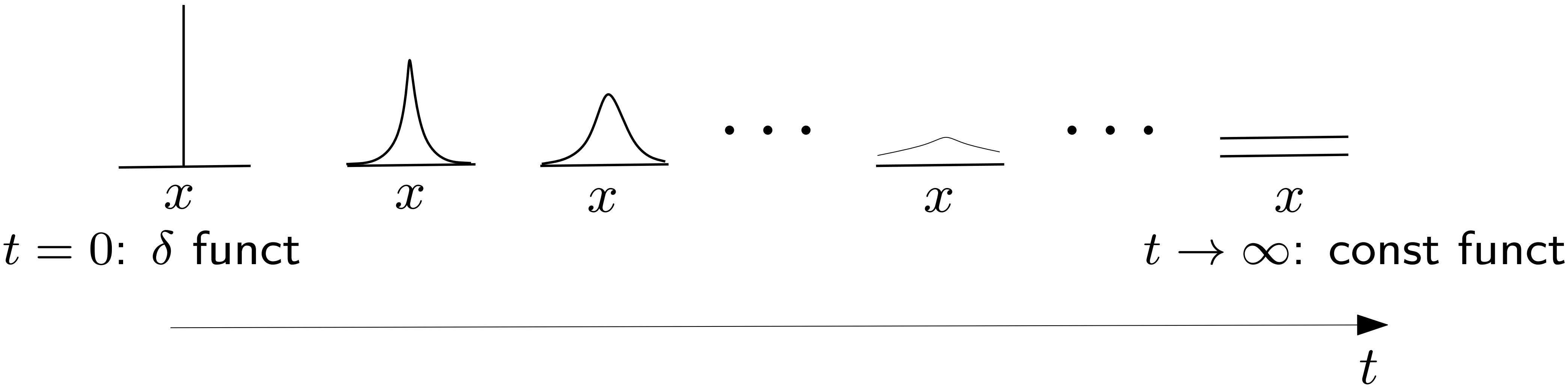

  - local features in small t's; global summaries in big t's

- however, $\{k_t(x,\cdot)\}_t$'s complexity is extremely high
  - difficult to compare $\{k_t(x,\cdot)\}_t$ with $\{k_t(x',\cdot)\}_t$

Heat Kernel Signature

- HKS is the restriction of $\{k_t(x, \cdot)\}_t$ to the temporal domain
  - $\mathsf{HKS}_x : \mathbb{R}^+ \to \mathbb{R}^+$ by $\mathsf{HKS}_x(t) = k_t(x, x)$

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  - concise and commensurable
  - multi-scale
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  - isometric invariant
  - informative?

Heat Kernel Signature

- $\{\text{HKS}_x\}_{x \in M}$ is informative

Informative Theorem. If the eigenvalues of M and N are not repeated, a homeomorphism $T : M \to N$ is isometric iff $k_t^M(x,x) = k_t^N(T(x), T(x))$ for any $x \in M$ and any $t > 0$.

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  - almost all shapes have no repeated eigenvalues [BU82]

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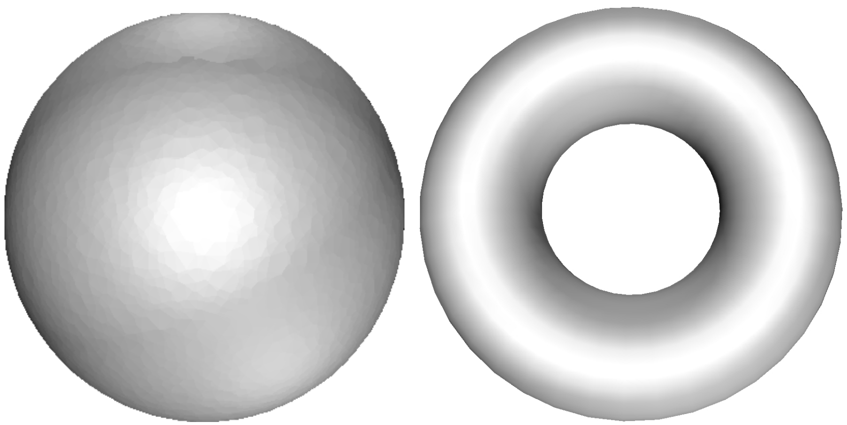
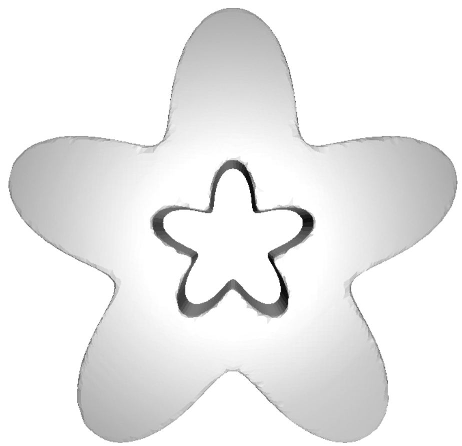
  - almost all shapes have no repeated eigenvalues [BU82]
  - the theorem fails if there are repeated eigenvalues



fails!

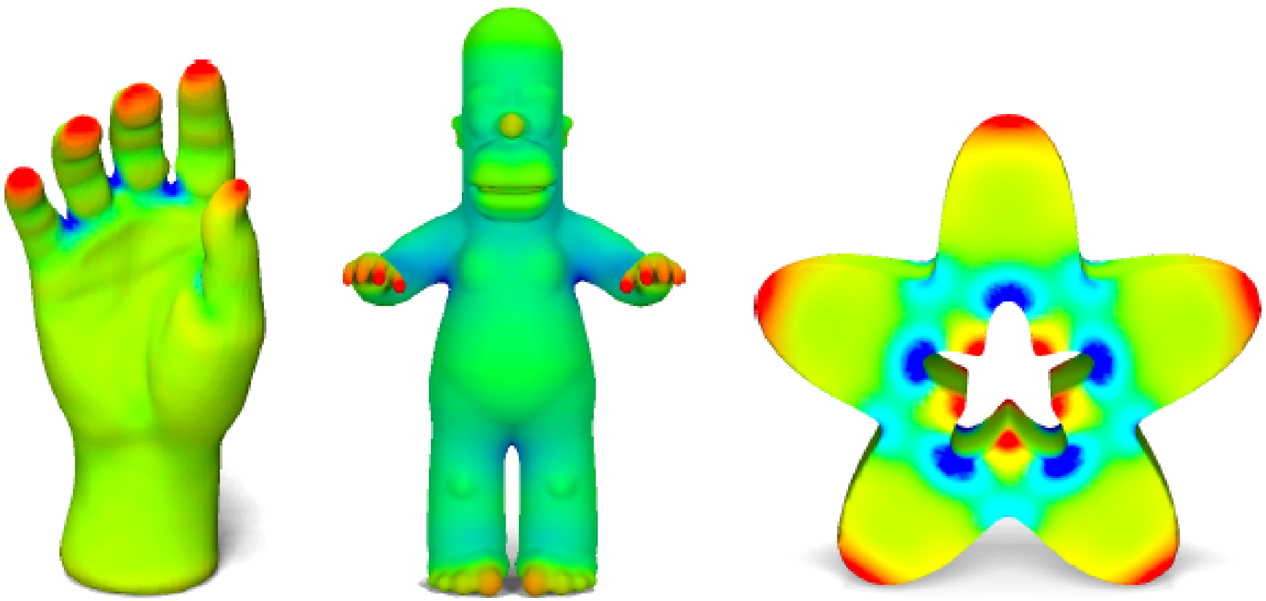


fails?

Relation to Curvature

- the polynomial expansion of HKS at small t:

$$\mathsf{HKS}_x(t) = k_t(x,x) = (4\pi t)^{-d/2}(1 + \frac{1}{6}s(x)t + O(t^2))$$



plot of $k_t(x,x)$ for a fixed t

Relation to Diffusion Distance

- diffusion distance [Laf04]

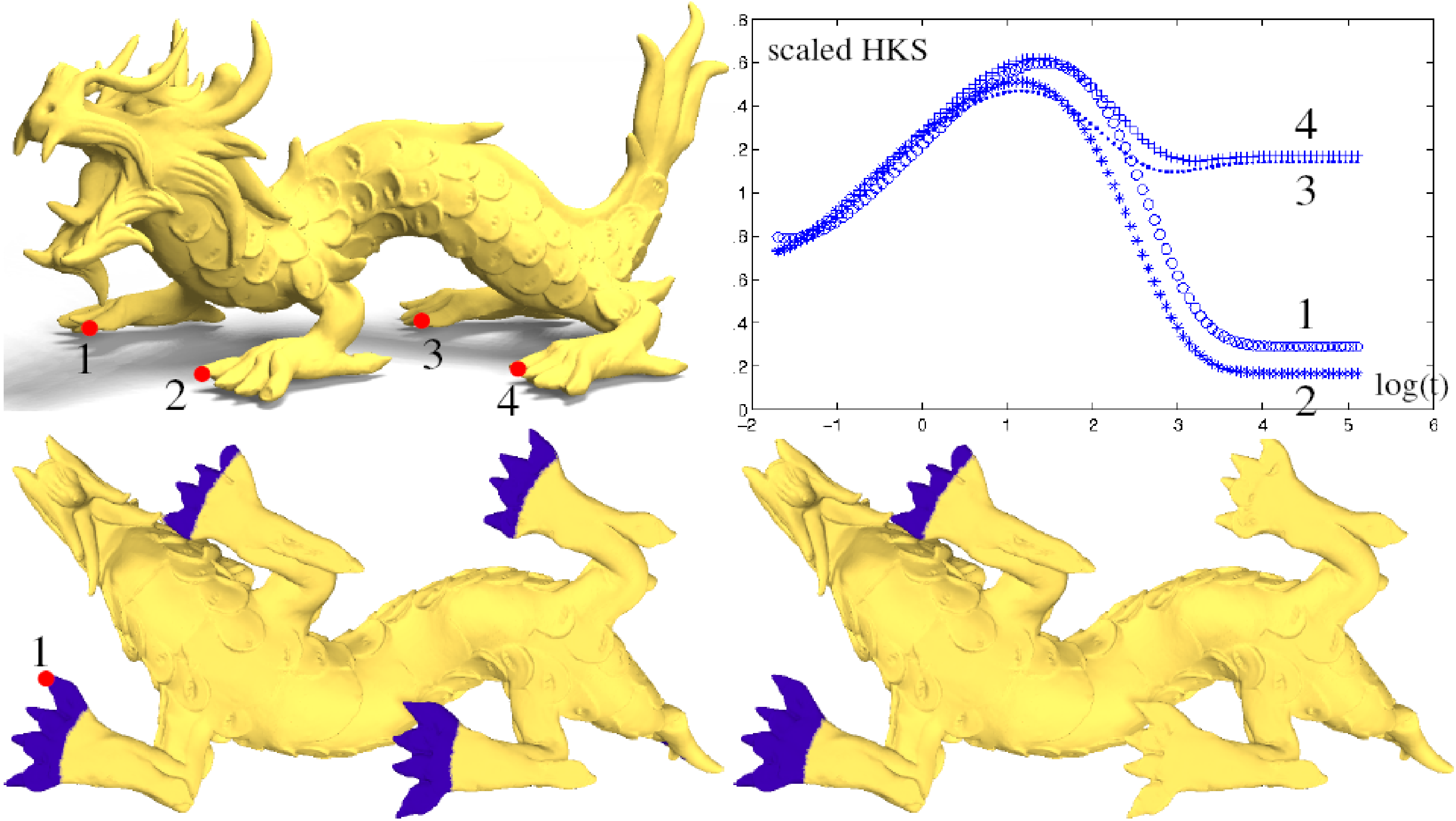
$$d_t^2(x,y) = k_t(x,x) + k_t(y,y) - 2k_t(x,y)$$

  - eccentricity in terms of diffusion distance

$$\begin{aligned} ecc_t(x) &= \frac{1}{A_M}\int_M d_t^2(x,y)dy \\ &= k_t(x,x) + H_M(t) - \frac{2}{A_M}, \end{aligned}$$

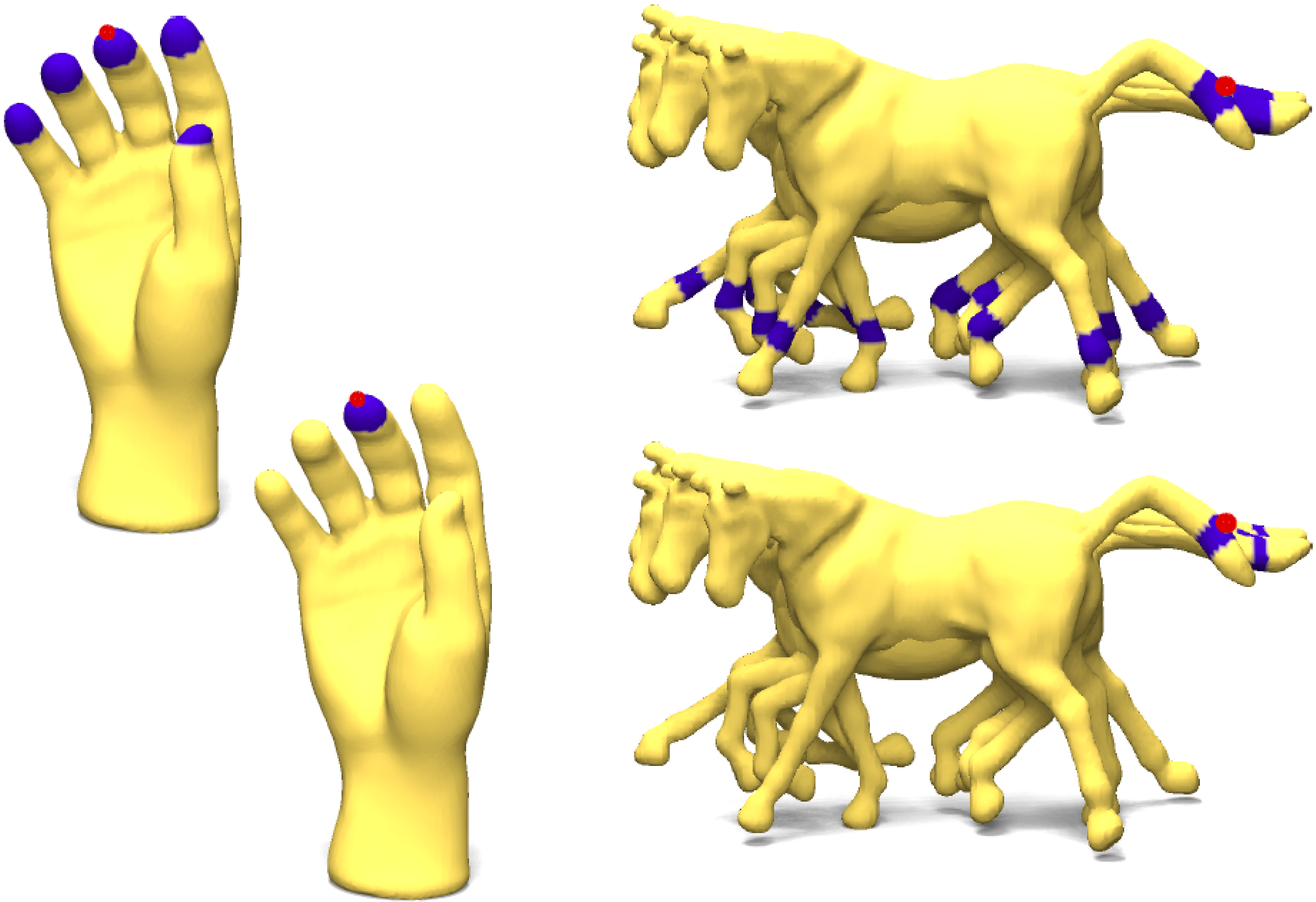
    - $ecc_t(x)$ and $k_t(x,x)$ have the same level sets, in particular, extrema points
    - shape segmentation [dGGV08]

Multi-Scale Matching



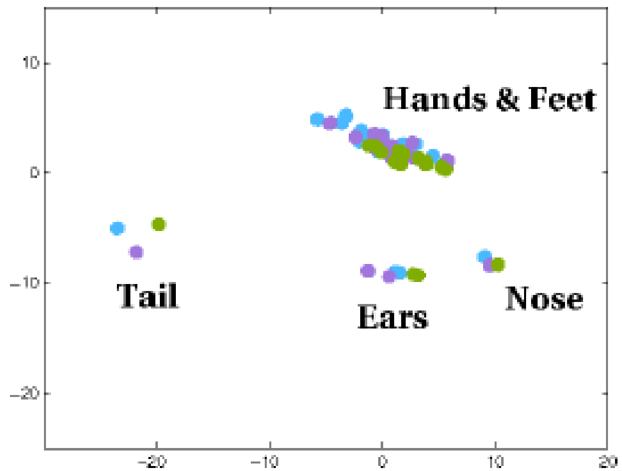
scaled HKS: $\frac{k_t(x,x)}{\int_M k_t(x,x)dx}$

Multi-Scale Matching

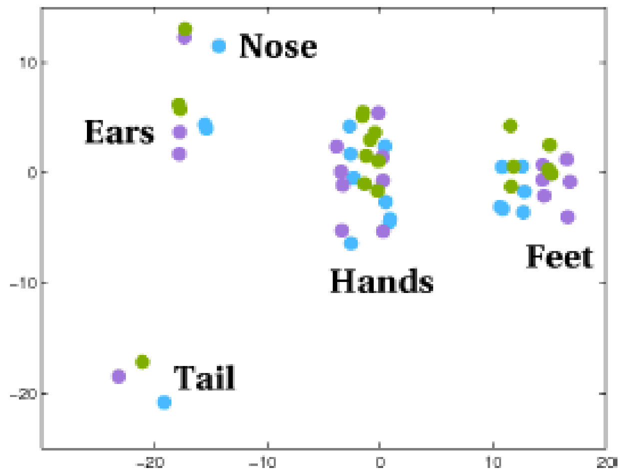


Multi-Scale Matching

(a) maxima of $k_t(x, x)$ for a fixed t.



(b) $t = [0.1,\ 4]$

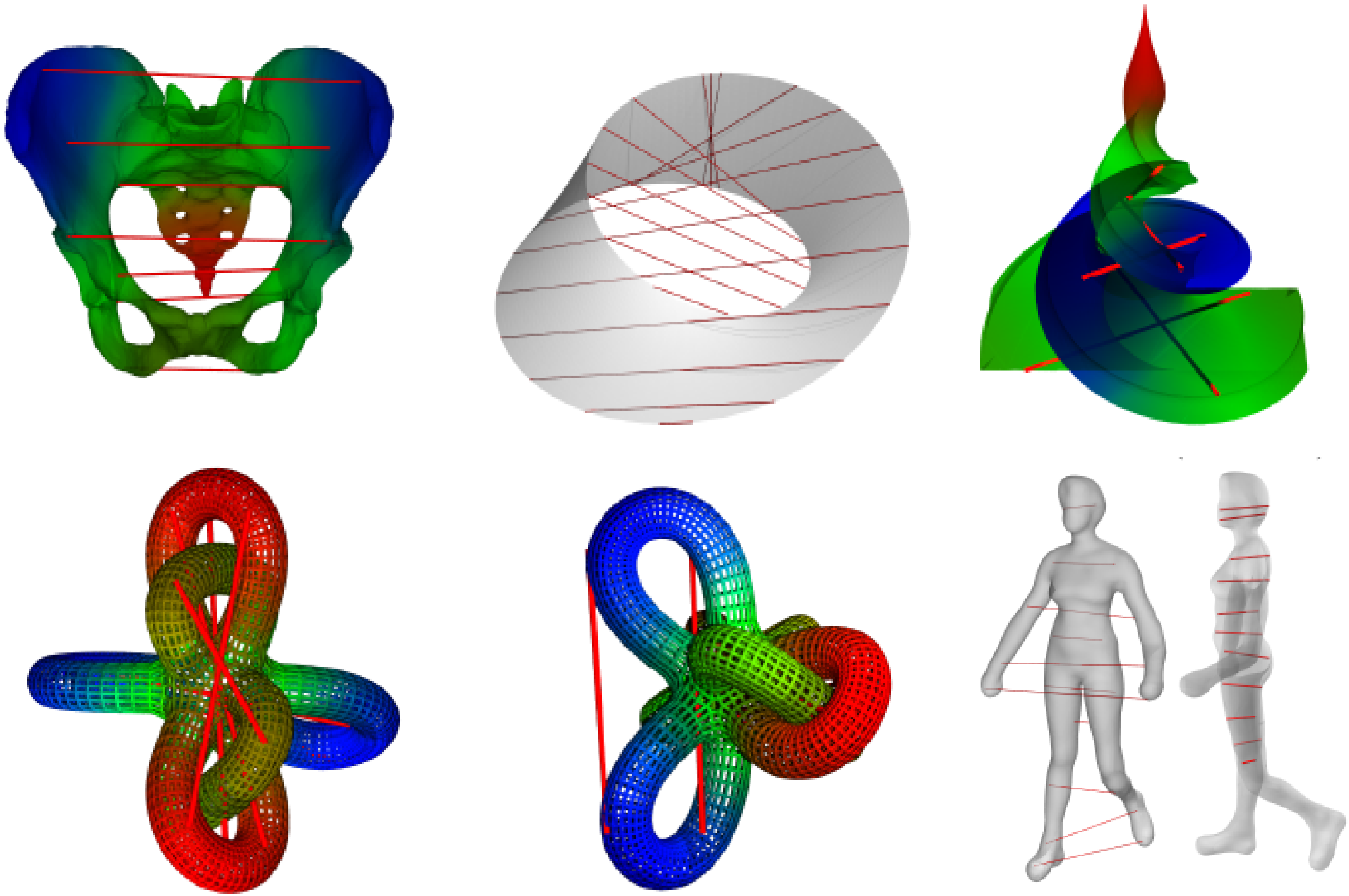


(c) $t = [0.1,\ 80]$

Thank you for your attention

Questions?

Results



Computation

- Laplace-Beltrami Operator:
  - based on its eigenfunctions and eigenvalues
    $k_t(x, y) = \sum_i e^{-\lambda_i t} \phi_i(x) \phi_i(y)$
    $\Rightarrow \quad \mathsf{HKS}_x(t) = k_t(x, x) = \sum_i e^{-\lambda_i t} \phi_i^2(x)$
- discrete case:
  - build the discrete Laplace operator $L = A^{-1} W$ [BSW08]
  - solve $W\phi = \lambda A \phi$
  - compute $\mathsf{HKS}_x(t) = \sum_i e^{-\lambda_i t} \phi_i^2(x)$

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    - third level bulletin