# PERRON FROBENIUS THEOREM FOR NONNEGATIVE TENSORS

K.C. CHANG, KELLY PEARSON, AND TAN ZHANG

Abstract. We generalize the Perron Frobenius Theorem for nonnegative matrices to the class of nonnegative tensors.

## 1. Introduction

Perron Frobenius Theorem is a fundamental result for nonnegative matrices. It has numerous applications, not only in many branches of mathematics, such as Markov chains, graph theory, game theory, and numerical analysis, but in various fields of science and technology, e.g. economics, operational research, and recently, page rank in the internet, as well. Its infinite dimensional extension is known as the Krein Rutman Theorem for positive linear compact operators, which has also been widely applied to Partial Differential Equations, Fixed Point Theory, and Functional Analysis.

In late studies of numerical multilinear algebra [7][4][1], eigenvalue problems for tensors have been brought to special attention. In particular, the Perron Frobenius Theorem for nonnegative tensors is related to measuring higher order connectivity in linked objects [5] and hypergraphs [6].

The purpose of this paper is to extend Perron Frobenius Theorem to nonnegative tensors.

It is well known that Perron Frobenius Theorem has the following two forms:

**Theorem 1.1.** *(Weak Form) If $A$ is a nonnegative square matrix, then*

1. *$r(A)$, the spectral radius of $A$, is an eigenvalue.*
2. *There exists a nonnegative vector $x_0 \neq 0$ such that*

$$Ax_0 = r(A)x_0. \tag{1.1}$$

We recall the following definition of irreducibility of $A$: a square matrix $A$ is said to be reducible if it can be placed into block upper-triangular form by simultaneous row/column permutations. A square matrix that is not reducible is said to be irreducible.

**Theorem 1.2.** *(Strong Form) If $A$ is an irreducible nonnegative square matrix, then*

1. *$r(A) > 0$ is an eigenvalue.*
2. *There exists a nonnegative vector $x_0 > 0$, i.e. all components of $x_0$ are positive, such that $Ax_0 = r(A)x_0$.*

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(3) *(Uniqueness) If $\lambda$ is an eigenvalue with a nonnegative eigenvector, then $\lambda = r(A)$.*
(4) *$r(A)$ is a simple eigenvalue of $A$.*
(5) *If $\lambda$ is an eigenvalue of $A$, the $|\lambda| \leq r(A)$.*

We shall extend these results to nonnegative tensors. But first, let us recall some definitions on tensors. An $m$-order $n$-dimensional tensor $\mathcal{C}$ is a set of $n^m$ real entries

$$\mathcal{C} = (c_{i_1 \cdots i_m}), \quad c_{i_1 \cdots i_m} \in \mathbb{R}, \quad 1 \leq i_1, \ldots, i_m \leq n. \tag{1.2}$$

$\mathcal{C}$ is called nonnegative (or respectively positive) if $c_{i_1 \cdots i_m} \geq 0$ (or respectively $c_{i_1 \cdots i_m} > 0$). To an $n$-vector $x = (x_1, \cdots, x_n)$, real or complex, we define an $n$-vector:

$$\mathcal{C}x^{m-1} := \left( \sum_{i_2, \ldots, i_m = 1}^{n} c_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} \right)_{1 \leq i \leq n}. \tag{1.3}$$

Suppose $\mathcal{C}x^{m-1} \neq 0$, a pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called an eigenvalue and an eigenvector, if they satisfy

$$\mathcal{C}x^{m-1} = \lambda x^{[m-1]}, \tag{1.4}$$

where $x^{[m-1]} = (x_1^{m-1}, \ldots, x_n^{m-1})$. When $m$ is even, and $\mathcal{C}$ is symmetric, this was introduced by Qi [7]; when $m$ is odd, Lim [4] used $(x_1^{m-1 \operatorname{sgn} x_1}, \ldots, x_n^{m-1} \operatorname{sgn} x_n)$ on the right-hand side instead, and the notion has been generalized in Chang Pearson Zhang [1].

Unlike matrices, the eigenvalue problem for tensors are nonlinear, namely, finding nontrivial solutions of polynomial systems in several variables. This feature enables us to employ different methods in generazations.

The main results of this paper are stated as follows:

**Theorem 1.3.** *If $\mathcal{A}$ is a nonnegative tensor of order $m$ dimension $n$, then there exist $\lambda_0 \geq 0$ and a nonnegative vector $x_0 \neq 0$ such that*

$$\mathcal{A}x^{m-1} = \lambda_0 x_0^{[m-1]}. \tag{1.5}$$

**Theorem 1.4.** *If $\mathcal{A}$ is an irreducible nonnegative tensor of order $m$ dimension $n$, then the pair $(\lambda_0, x_0)$ in equation (1.5) satisfy:*

(1) *$\lambda_0 > 0$ is an eigenvalue.*
(2) *$x_0 > 0$, i.e. all components of $x_0$ are positive.*
(3) *If $\lambda$ is an eigenvalue with nonnegative eigenvector, then $\lambda = \lambda_0$. Moreover, the nonnegative eigenvector is unique up to a multiplicative constant.*
(4) *If $\lambda$ is an eigenvalue of $A$, then $|\lambda| \leq \lambda_0$.*

However, unlike matrices, such $\lambda_0$ is not necessarily a simple eigenvalue for tensors in general. We shall present an example to demostrate such distinction. Furthermore, some additional conditions will be imposed to ensure the simplicity of the eigenvalue $\lambda_0$.

In the paper of Lim [4], some of the above conclusions in Theorem 1.4 were obtained. However, we shall study this problem more systematically in a more self-contained manner via a different approach here.

We organize our paper as follows: §2 is devoted to prove the main theorems, except (4) of Theorem 1.4. In §3, we discuss the simplicity of $\lambda_0$. In §4, we study an extended Collatz's minimax Theorem, from which assertion (4) of Theorem 1.4 will follow as a direct consequence. In the last §5, various extensions of the main results will be given.

## 2. Proofs of the main theorems

Let $X = \mathbb{R}^n$. It has a positive cone $P = \{(x_1, \ldots, x_n) \in X \mid x_i \geq 0, 1 \leq i \leq n\}$. The interior of $P$ is denoted $\operatorname{int} P = \{(x_1, \ldots, x_n) \in P \mid x_i > 0, 1 \leq i \leq n\}$. An order is induced by $P$: $\forall x, y \in X$, we define $x \leq y$ if $y - x \in P$, and $x < y$ if $x \leq y$ and $x \neq y$.

A $m$ order tensor $\mathcal{C}$ is hence associated with a nonlinear $(m-1)$ homogeneous operator $C : X \to X$ by $Cx = \mathcal{C}x^{m-1}, \forall x \in X$, i.e.,

$$C(tx) = t^{m-1}Cx, \forall x \in X, \forall t \in R^1. \tag{2.1}$$

It is obviously seen that if $\mathcal{C}$ is nonnegative (or respectively positive), i.e, all entries are nonnegative (or respectively positive), then the associate nonlinear operator $C : P \to P$ (or $C : P \setminus \{0\} \to \operatorname{int} P$). Moreover, if $\mathcal{C}$ is nonnegative, then

$$Cx \leq Cy, \forall x \leq y, \forall x, y \in P. \tag{2.2}$$

And we are now ready for the proof of Theorem 1.3:

*Proof.* We reduce the problem to a fixed point problem as follows. Let $D = \{(x_1, \ldots, x_n) \in X \mid x_i \geq 0, 1 \leq i \leq n, \sum_{i=1}^n x_i = 1\}$ be a closed convex set. One may assume $\mathcal{A}x^{m-1} \neq 0 \ \forall x \in D$. For otherwise, there exists at least a $x_0 \in D$ so that $\mathcal{A}x_0^{m-1} = 0$. Let $\lambda_0 = 0$, then $(\lambda_0, x_0)$ is a solution to (1.3), and we are done. Then the following map $F : D \to D$ is well defined:

$$F(x)_i = \frac{(\mathcal{A}x^{m-1})_i^{\frac{1}{m-1}}}{\sum_{j=1}^n (\mathcal{A}x^{m-1})_j^{\frac{1}{m-1}}}, \qquad 1 \leq i \leq n, \tag{2.3}$$

where $(\mathcal{A}x^{m-1})_i$ is the $i-th$ component of $\mathcal{A}x^{m-1}$. $F : D \to D$ is clearly continuous. According to the Brouwer's Fixed Point Theorem, $\exists\, x_0 \in D$ such that $F(x_0) = x_0$, i.e.

$$\mathcal{A}x_0^{m-1} = \lambda_0 x_0^{[m-1]},$$

where

$$\lambda_0 = \left( \sum_{j=1}^n (\mathcal{A}x_0^{m-1})_j^{\frac{1}{m-1}} \right)^{m-1}. \tag{2.4}$$

□

We now turn to Theorem 1.4. If $\mathcal{A}$ is positive then we can use similar arguments used in positive matrices to establish conclusions (1) - (3) in Theorem 1.4 based on Theorem 1.3.

Our purpose in the remaining of this section is to introduce a condition on tensors which lies in between positivity and nonnegativity to ensure similar results hold as Perron Frobenius Theorem for matrices.

**Definition 2.1.** (Reducibility) A tensor $\mathcal{C} = (c_{i_1 \cdots i_m})$ of order $m$ dimension $n$ is called reducible, if there exists a nonempty proper index subset $I \subset \{1, \ldots, n\}$ such that

$$c_{i_1 \cdots i_m} = 0, \quad \forall i_1 \in I, \quad \forall i_2, \ldots, i_m \notin I.$$

If $\mathcal{C}$ is not reducible, then we call $\mathcal{C}$ irreducible.

**Lemma 2.2.** If a nonnegative tensor $\mathcal{C}$ of order $m$ dimension $n$ is irreducible, then

$$\sum_{i_2,\ldots,i_m=1}^{n} c_{ii_2\cdots i_m} > 0, \quad \forall 1 \le i \le n.$$

*Proof.* Suppose not, then there exists $i_0$ so that $\sum_{i_2,\ldots,i_m=1}^{n} c_{i_0 i_2\cdots i_m} = 0$. Since $\mathcal{C}$ is nonnegative, $c_{i_0 i_2\cdots i_m} = 0 \ \forall i_2,\ldots,i_m$. In particular, if we let $I = \{i_0\}$, then $c_{i_1 i_2\cdots i_m} = 0, \forall i_1 \in I$ and $\forall i_2,\ldots,i_m \notin I$, this contradicts irreducibility. $\square$

We are now ready for the proof of Theorem 1.4.

*Proof.* 1° First, we prove $x_0 \in \operatorname{int} P$. Note $P \setminus \operatorname{int} P = \partial P = \cup_{I\in\Lambda} F_I$, where $\Lambda$ is the set of all index subsets $I$ of $\{1,\ldots,n\}$ and

$$F_I = \{(x_1,\ldots,x_n) \in P \mid x_i = 0 \, \forall i \in I, \text{and}\, x_j \ne 0 \, \forall j \notin I\}.$$

Suppose $x_0 \notin \operatorname{int} P$, since $x_0 \neq 0$, there must be a maximal proper index subset $I \in \Lambda$ such that $x_0 \in F_I$, i.e. $(x_0)_i = 0 \ \forall i \in I$ and $(x_0)_j > 0 \ \forall j \notin I$. Let $\delta = \operatorname{Min}\{(x_0)_j \mid j \notin I\}$, we then have $\delta > 0$. Since $x_0$ is an eigenvector, $Ax_0 \in F_I$, i.e.

$$\sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\cdots i_m}(x_0)_{i_2}\cdots(x_0)_{i_m} = 0, \quad \forall i \in I.$$

It follows

$$\delta^{m-1} \sum_{i_2,\ldots,i_m\notin I} a_{ii_2\cdots i_m} \le \sum_{i_2,\ldots,i_m\notin I} a_{ii_2\cdots i_m}(x_0)_{i_2}\cdots(x_0)_{i_m} = 0, \quad \forall i \in I,$$

hence we have $a_{ii_2\cdots i_m} = 0 \quad \forall i \in I, \quad \forall i_2,\ldots,i_m \notin I$, i.e. $\mathcal{A}$ is reducible, a contradiction.
2° Combining 1° and Lemma 2.2, we have $\lambda_0 > 0$.
3° We now prove the eigenvalue corresponding to the positive eigenvector is unique, namely, if $(\lambda, x)$ and $(\mu, y) \in \mathbb{R} \times P$ are solutions of (1.5), then $\lambda = \mu$. According to 1° and 2°, such $x, y \in \operatorname{int} P$ and $\lambda, \mu > 0$. $\forall z \in \operatorname{int} P$ and $\forall w \notin P$, we define $\delta_z(w) = \{s \in \mathbb{R}_+ \mid z + sw \in P\}$, then $\delta_z(w) > 0$, $z + tw \in P$ for $0 \le t \le \delta_z(w)$ and $z + tw \in P$ for $t > \delta_z(w)$. Applying these to $(z, w) = (x, -y)$, we have $x - ty \in P$ for $0 \le t \le \delta_z(-y)$. By definition and (2.1), (2.2),

$$\lambda x^{[m-1]} = \mathcal{A}x^{m-1} \ge \delta_x(-y)^{m-1}\mathcal{A}y^{m-1} = \mu\delta_x(-y)^{m-1}y^{[m-1]}, \tag{2.5}$$

it follows $x \ge (\frac{\mu}{\lambda})^{\frac{1}{m-1}}\delta_x(-y)y$, thus $\mu \le \lambda$.
Likewise, if we interchange $x$ and $y$, it follows $y \ge (\frac{\lambda}{\mu})^{\frac{1}{m-1}}\delta_y(-x)x$, and thus $\lambda \le \mu$. We have hence proved $\lambda = \mu$. Therefore, the only eigenvalue corresponding to the positive eigenvector is $\lambda_0$.
4° We prove the positive eigenvector is unique up to a multiplicative constant, i.e. if $x_0, x \in P \setminus \{0\}$ satisfying $\mathcal{A}x_0^{m-1} = \lambda_0 x_0^{[m-1]}$ and $\mathcal{A}x^{m-1} = \lambda_0 x^{[m-1]}$, then $x = kx_0$ for some constant $k$. It has been known that $x_0 \in \operatorname{int} P$, by the definition of $\delta_{x_0}(-x)$, we have $x_0 - tx \in P$ for $0 \le t \le \delta_{x_0}(-x)$ and $x_0 - tx \notin P$ for $t > \delta_{x_0}(-x)$. This implies $x_0 - t_0 x \in \partial P$, where $t_0 = \delta_{x_0}(-x)$. So there exists a nonempty maximal index subset $I \subset \{1,\ldots,n\}$ such that $x_0 - t_0 x \in F_I$. If $I = \{1,\ldots,n\}$,

then $x_0 = t_0 x$, and we are done. Otherwise, $I$ is a nonempty proper subset. There exist $\epsilon > 0$ and $\delta > 0$ such that

$$\begin{aligned} &(x_0)_i \geq \delta, \quad \forall i \in \{1, 2, ....n\}, \\ &0 < t_0 x_i = (x_0)_i, \qquad \forall i \in I, \\ &0 < \frac{t_0 x_i}{(x_0)_i} < 1 - \epsilon \quad \forall i \notin I, \end{aligned}$$

and then $\forall i \in I$

$$\sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\cdots i_m}[(x_0)_{i_2}\cdots(x_0)_{i_m} - t_0^{m-1}x_{i_2}\cdots x_{i_m}] = \lambda_0[(x_0)_i^{m-1} - (t_0 x_i)^{m-1}] = 0.$$

We have

$$t_0^{m-1}x_{i_2}\cdots x_{i_m} \leq (x_0)_{i_2}\cdots(x_0)_{i_m} \quad \forall i_2, \ldots, i_m,$$

and

$$t_0^{m-1}x_{i_2}\cdots x_{i_m} \leq (1-\epsilon)^{m-1}(x_0)_{i_2}\cdots(x_0)_{i_m} \quad \forall i_2, \ldots, i_m \notin I.$$

It follows

$$\begin{aligned} &\delta^{m-1}(1-(1-\epsilon)^{m-1}) \sum_{i_2,\ldots,i_m \notin I} a_{ii_2\cdots i_m} \\ &\leq \sum_{i_2,\ldots,i_m \notin I} a_{ii_2\cdots i_m}[(x_0)_{i_2}\cdots(x_0)_{i_m} - t_0^{m-1}x_{i_2}\cdots x_{i_m}] \\ &\leq \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\cdots i_m}[(x_0)_{i_2}\cdots(x_0)_{i_m} - t_0^{m-1}x_{i_2}\cdots x_{i_m}] = 0 \quad \forall i \in I, \end{aligned}$$

thus $a_{ii_2\cdots i_m} = 0 \quad \forall i \in I, \quad \forall i_2, \ldots, i_m \notin I$, i.e. $\mathcal{A}$ is reducible, a contradiction. □

**Remark:** By the same argument used in 1° of the proof of Theorem 1.4, the following improvement also holds: Assume $\mathcal{A}$ is an irreducible nonnegative tensor. If $x_0 \in P \setminus \{0\}$ is a solution of the inequality $\mathcal{A}x^{m-1} \leq \lambda x^{[m-1]}$, then $x_0 \in \operatorname{int} P$.

## 3. The simplicity of the eigenvalue $\lambda_0$

For a matrix (i.e. $m = 2$) $A$, an eigenvalue $\lambda$ is called algebraically simple, if $\lambda$ is a simple root of the characteristic polynomial $\det(A - \lambda I)$, and is called geometrically simple if $\dim \operatorname{Ker}(A - \lambda I) = 1$. We will generalize these notions to the tensor setting. Since the operator $A$ associate with a tensor $\mathcal{A}$ is nonlinear but homogeneous, we can define the geometric multiplicity of an eigenvalue of $\mathcal{A}$ as follow:

**Definition 3.1.** Let $\lambda$ be an eigenvalue of

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}. \tag{3.1}$$

We say $\lambda$ has geometric multiplicity $q$, if the maximum number of linearly independent eigenvectors corresponding to $\lambda$ equals $q$. If $q = 1$, then $\lambda$ is called geometrically simple.

It is worth noting the geometric multiplicity for a real eigenvalue $\lambda$ of a real matrix $A$ is independent to the field over the vector space being real or complex, i.e.,

$$dim_{\mathbf{R}}\{x \in \mathbf{R}^n | (A - \lambda I)x = 0\} = dim_{\mathbf{C}}\{z \in \mathbf{C}^n | (A - \lambda I)z = 0\}.$$

This is due to the fact that if $z = x + iy \in \mathbf{R}^n + i\mathbf{R}^n$ satisfies $(A - \lambda I)z = 0$, then both $x, y \in Ker(A - \lambda I) \bigcap \mathbf{R}^n$.

As to higher order tensors, since $\mathcal{A}x^{m-1}$ is $m-1$ homogeneous, we still have

real geometric multiplicity $\leq$ complex geometric multiplicity,

but not equal in general. This can be seen from the following example:

**Example 3.2.** Let $m = 3$ and $n = 2$. Consider $\mathcal{A} = (a_{ijk})$ where $a_{111} = a_{222} = 1$, $a_{122} = a_{211} = \epsilon$ for $0 < \epsilon < 1$, and $a_{ijk} = 0$ for other $(ijk)$. Then the eigenvalue problem becomes:

$$\begin{cases} x_1^2 + \epsilon x_2^2 = \lambda x_1^2 \\ \epsilon x_1^2 + x_2^2 = \lambda x_2^2. \end{cases} \tag{3.2}$$

We have $\lambda = 1 + \varepsilon$, with eigenvectors: $u_1 = (1, 1)$ and $u_2 = (1, -1)$, and $\lambda = 1 - \varepsilon$ with eigenvectors: $u_3 = (1, i)$, and $u_4 = (1, -i)$.
In this example we see that

real geometric multiplicity of $\lambda = 1 + \varepsilon =$ complex geometric multiplicity $= 2$,
and

real geometric multiplicity of $\lambda = 1 - \varepsilon$ is 0, and complex geometric multiplicity is 2.

The same example also shows the nonnegative irreducible tensor $\mathcal{A}$ has a positive eigenvalue $1 + \varepsilon$ with unique positive eigenvector (up to a multiplicative constant), which is not geometrically simple neither in $\mathbf{R}$ nor in $\mathbf{C}$.

**Example 3.3.** Let $m = 4, n = 2, \mathcal{A} = (a_{ijkl})$ with $a_{1222} = a_{2111} = 1$ and $a_{ijkl} = 0$ elsewhere. Then after computation, we see there are two eigenvalues: $\lambda = \pm 1$, with eigenvectors: $(x, \pm x), (x, \pm \exp \frac{2\pi i}{3} x), (x, \pm \exp \frac{4\pi i}{3} x)$. Therefore both $\lambda = \pm 1$ are all real geometrically simple, but with complex geometrical multiplicity 3.

In the following, we shall seek a sufficient condition to ensure the real geometric simplicity of $\lambda_0$.

In case $m$ is odd, there are two different types of eigenvalue problems, which impose the same constraints on $P$:

(1) $\mathcal{A}x^{m-1} = \lambda(x_1^{m-1}, \ldots, x_n^{m-1})$,
(2) $\mathcal{A}x^{m-1} = \lambda(\operatorname{sgn} x_1 x_1^{m-1}, \ldots, \operatorname{sgn} x_n x_n^{m-1})$.

**Theorem 3.4.** *Let $m$ be odd, and let $\mathcal{A}$ be an irreducible nonnegative tensor of order $m$ dimension $n$. If $\mathcal{A}x^{m-1}$ is invariant under any one of the transformations: $(x_1, ..., x_n) \to (\pm x_1, ..., \pm x_n)$, except the identity and its reflection, then $\lambda_0$ is not geometrically simple for problem (1). If all terms in $\mathcal{A}x^{m-1}$ are monomials of $x_1^2, \ldots, x_n^2$, i.e. $a_{i_1 i_2 \cdots i_m} \neq 0$ only if the numbers of indices appearing in $\{i_2, \ldots, i_m\}$ are all even, $\forall i_1$, then $\lambda_0$ is real geometrically simple for problem (2).*

*Proof.* (1) Let $T$ be the transformation, to which $\mathcal{A}x^{m-1}$ is invariant under. By assumption, if $x_0 = (x_1^0, \ldots, x_n^0) \in \operatorname{int} P$ is a solution of (1), then $Tx_0$ is also a solution of (1) corresponding to the same eigenvalue $\lambda_0$, so $\lambda_0$ is not geometrically simple.
(2) By the assumption, $\mathcal{A}x^{m-1} \geq 0, \forall x \in R^n$, which implies all solutions of (2) must be in $P$. Using assertion (3) of Theorem 1.4, we see $x = kx_0$, i.e. $\lambda_0$ is real geometrically simple. $\square$

We next examine the case when $m$ is even. We introduce a condition on $\mathcal{C}$ to ensure the associated nonlinear operator $C$ is increasing, i.e.

$$x \leq y \Rightarrow Cx \leq Cy. \tag{3.3}$$

Comparing with (2.2), there is no restriction: $x, y \in P$ in (3.3).

**Definition 3.5.** (Condition (M)) A tensor $\mathcal{C} = c_{i_1 i_2 \cdots i_m}$ of order $m > 2$ dimension $n$ is said to satisfy Condition (M), if there exists a nonnegative matrix $D = (d_{ij})$ such that $c_{i_1 i_2 \cdots i_m} = d_{i_1 i_2} \delta_{i_2 \cdots i_m}$, where $\delta_{i_2 \cdots i_m}$ is the Kronecker delta.

**Remark:** For $m = 2$, Condition (M) is trivial, hence is superfluous.

In fact, if $m$ is even, Condition (M) on $\mathcal{C}$ implies

$$\frac{\partial}{\partial x_j} (\mathcal{C} x^{m-1})_i = (m-1) \sum_{j=1}^{n} d_{ij} x_j^{m-2} \geq 0 \ \forall i, j,$$

and then $Cx \leq Cy, \forall x \leq y$, $\forall x, y \in R^n$. We now state and prove the following:

**Theorem 3.6.** *Let $m$ be even, and let $\mathcal{A}$ be an irreducible nonnegative tensor. If $\mathcal{A}$ satisfies Condition (M), then the eigenvalue $\lambda_0$ for nonnegative eigenvector is real geometrically simple.*

**Remark:** To the special problem, it can, by setting $y = x^{[m-1]}$, be reduced to the problem for matrices, hence becomes a direct consequence of Perron Frobenius Theorem. However, we present the following proof since it will be useful for more general problems, see §5.

*Proof.* We follow $4^\circ$ in the proof of Theorem 1.4. We note the only difference is now $x \in \mathbb{R}^n \setminus \{0\}$ but not $P \setminus \{0\}$. We still have $t_0 = \delta_{x_0}(-x)$ such that $x_0 - tx \in P$ for $0 \leq t \leq t_0$, and $x_0 - tx \notin P$ for $t > t_0$. We want to show $x_0 = t_0 x$. Suppose not, one has $(x_0)_i \geq \delta > 0, \forall i$ and a nonempty proper index subset $I$ such that $t_0 x_i = (x_0)_i \quad \forall i \in I$ and $t_0 x_i < (1-\epsilon)(x_0)_i \quad \forall i \notin I$. It follows $\forall i \in I$

$$\begin{aligned} & \delta^{m-1}(1 - (1-\epsilon)^{m-1}) \sum_{j \notin I} a_{ij \cdots j} \\ \leq & \sum_{j \notin I} a_{ij \cdots j} [(x_0)_j^{m-1} - (t_0 x_i)^{m-1}] \\ \leq & \sum_{i_2, \ldots, i_m = 1}^{n} a_{i i_2 \cdots i_m} [(x_0)_{i_2} \cdots (x_0)_{i_m} - t_0^{m-1} x_{i_2} \cdots x_{i_m}] = 0, \end{aligned}$$

thus $a_{ij \cdots j} = 0 \quad \forall j \notin I$. Combining this with Condition (M), we obtain $a_{i_1 i_2 \cdots i_m} = 0 \quad \forall i_1 \in I, \quad \forall i_2, \ldots, i_m \notin I$, which contradicts the irreducibility of $\mathcal{A}$. Therefore, $x_0 = t_0 x$, i.e. $\lambda_0$ is geometrically simple as desired. □

We next define the algebraic simplicity of the eigenvalue of (3.1). We follow the approach described in Cox et al. [pp. 97 – 105] to define the characteristic polynomial $\psi_{\mathcal{A}}(\lambda)$ of $\mathcal{A}$ by

$$\psi_{\mathcal{A}}(\lambda) := Res((\mathcal{A} x^{m-1})_1 - \lambda x_1^{m-1}, \ldots, (\mathcal{A} x^{m-1})_n - \lambda x_n^{m-1})),$$

where $Res(P_1, \ldots, P_n)$ is the resultant of n homogeneous polynomials $P_1, \ldots, P_n$. For each $\mathcal{A}$, such $\psi_{\mathcal{A}}(\lambda)$ is unique up to an extraneous factor.

**Definition 3.7.** Let $\lambda$ be an eigenvalue of (3.1). We say $\lambda$ has algebraic multiplicity $p$, if $\lambda$ is a root of $\psi_{\mathcal{A}}(\lambda)$ of multiplicity $p$. And we call $\lambda$ an algebraically simple eigenvalue, if $p = 1$.

To the Example 3.2, we have known that $\lambda = 1 + \epsilon$ has geometric multiplicity 2 both in real or in complex fields. After computation we have

$$\begin{aligned}\psi_{\mathcal{A}}(\lambda) &= \det \begin{pmatrix} 1-\lambda & 0 & \epsilon & 0 \\ 0 & 1-\lambda & 0 & \epsilon \\ \epsilon & 0 & 1-\lambda & 0 \\ 0 & \epsilon & 0 & 1-\lambda \end{pmatrix} \\ &= (\lambda - 1 + \epsilon)^2 (\lambda - 1 - \epsilon)^2,\end{aligned}$$

which shows the eigenvalue $\lambda_0 = 1 + \epsilon$ also has algebraic multiplicity 2.
By definition, we see

complex geometric multiplicity $\leq$ algebraic multiplicity,

but not equal in general, this can be seen in the next example.

**Example 3.8.** Let $m = 4$ and $n = 2$. Consider $\mathcal{A} = (a_{ijkl})$ where $a_{1111} = a_{1112} = a_{2122} = a_{2222} = 1$, and $a_{ijkl} = 0$ for other $(ijkl)$. Then the eigenvalue problem becomes:

$$\begin{cases} x_1^3 + x_1^2 x_2 = \lambda x_1^3 \\ x_1 x_2^2 + x_2^3 = \lambda x_2^3. \end{cases}$$

We compute to see

$$\begin{aligned}\psi_{\mathcal{A}}(\lambda) &= \det \begin{pmatrix} 1-\lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & 1-\lambda & 1 & 0 & 0 & 0 \\ 0 & 0 & 1-\lambda & 1 & 0 & 0 \\ 0 & 0 & 1 & 1-\lambda & 0 & 0 \\ 0 & 0 & 0 & 1 & 1-\lambda & 0 \\ 0 & 0 & 0 & 0 & 1 & 1-\lambda \end{pmatrix} \\ &= \lambda(\lambda - 2)(\lambda - 1)^4,\end{aligned}$$

which shows the eigenvalues $\lambda = 0, 2$ are all algebraically and geometrically simple, with eigenvectors $u_1 = (1, -1)$, and $u_2 = (1, 1)$ respectively, while $\lambda = 1$ has algebraic multiplicity 4. but has only two linearly independent eigenvectors $u_3 = (1, 0)$ and $u_4 = (0, 1)$, so its geometric multiplicity is 2.

## 4. A Minimax Theorem

The following well known [3] minimax theorem for irreducible nonnegative matrices will be extended to irreducible nonnegative tensors.

**Theorem 4.1.** *(Collatz) Assume that $A$ is an irreducible nonnegative $n \times n$ matrix, then*

$$\mathrm{Min}_{x \in \mathrm{int}\, P}\, \mathrm{Max}_{x_i > 0} \frac{(Ax)_i}{x_i} = \lambda_0 = \mathrm{Max}_{x \in \mathrm{int}\, P}\, \mathrm{Min}_{x_i > 0} \frac{(Ax)_i}{x_i}, \tag{4.1}$$

*where $\lambda_0$ is the unique positive eigenvalue corresponding to the positive eigenvector.*

In the remainder of this section, we will prove the following

**Theorem 4.2.** *Assume that $\mathcal{A}$ is an irreducible nonnegative tensor of order $m$ dimension $n$, then*

$$\mathrm{Min}_{x\in \mathrm{int}\, P}\, \mathrm{Max}_{x_i>0}\, \frac{(\mathcal{A}x^{m-1})_i}{x_i^{m-1}} = \lambda_0 = \mathrm{Max}_{x\in \mathrm{int}\, P}\, \mathrm{Min}_{x_i>0}\, \frac{(\mathcal{A}x^{m-1})_i}{x_i^{m-1}}, \tag{4.2}$$

*where $\lambda_0$ is the unique positive eigenvalue corresponding to the positive eigenvector.*

Before we proceed with the proof of Theorem 4.2, we first define the following two functions on $P \setminus \{0\}$:

$$\mu_*(x) = \mathrm{Min}_{x_i>0}\, \frac{(\mathcal{A}x^{m-1})_i}{x_i^{m-1}} \quad \text{and} \quad \mu^*(x) = \mathrm{Max}_{x_i>0}\, \frac{(\mathcal{A}x^{m-1})_i}{x_i^{m-1}}.$$

Clearly, $\mu_*(x) \le \mu^*(x)$.
Note $\mu^*(x)$ may be $+\infty$ on the boundary $\partial P \setminus \{0\}$.
Since both $\mu_*(x)$ and $\mu^*(x)$ are positive 0-homogeneous functions, we can restrict them on the compact set

$$\Delta = \{(x = (x_1, \ldots, x_n) \in P \mid \sum_{i=1}^{n} x_i = 1\}.$$

Now, $\mu_*$ is continuous and bounded from above and $\mu^*$ is continuous on $\Delta \bigcap \mathrm{int}\, P$ and is bounded from below, there exist $x_*, x^* \in \Delta$ such that

$$\begin{aligned} r_* &:= \mu_*(x_*) = \mathrm{Max}_{x\in\Delta}\, \mu_*(x) = \mathrm{Max}_{x\in P\setminus\{0\}}\, \mu_*(x), \\ r^* &:= \mu^*(x^*) = \mathrm{Min}_{x\in\Delta}\, \mu_*(x) = \mathrm{Min}_{x\in P\setminus\{0\}}\, \mu_*(x). \end{aligned}$$

Let $(\lambda_0, x_0) \in \mathbb{R}_+ \times \mathrm{int}\, P$ be the positive eigen-pair obtained in Theorem 1.4, we then have:

$$\mu^*(x^*) \le \mu^*(x_0) = \lambda_0 = \mu_*(x_0) \le \mu_*(x_*).$$

Therefore,

$$r^* \le \lambda_0 \le r_*. \tag{4.3}$$

We shall prove they are indeed all equal. To do so, we modify 3° in the proof of Theorem 1.4 as follows:

**Lemma 4.3.** Let $\mathcal{A}$ an irreducible nonnegative tensor of order $m$ dimension $n$. If $(\lambda, x)$ and $(\mu, y) \in \mathbb{R}_+ \times (P \setminus \{0\})$ satisfy $\mathcal{A}x^{m-1} = \lambda x^{[m-1]}$ and $\mathcal{A}y^{m-1} \ge \mu y^{[m-1]}$ (or respectively $\mathcal{A}y^{m-1} \le \mu y^{[m-1]}$), then $\mu \le \lambda$ (or respectively $\lambda \le \mu$).

*Proof.* We first assume $\mathcal{A}y^{m-1} \ge \mu y^{[m-1]}$. Since $x \in \mathrm{int}\, P$, we have $t_0 = \delta_x(-y) > 0$ such that $x - ty \in P$ for $0 \le t \le t_0$ and $x - ty \notin P$ for $t > t_0$. It implies:

$$\lambda x^{[m-1]} = \mathcal{A}x^{m-1} \ge t_0^{m-1} \mathcal{A}y^{m-1} \ge t_0^{m-1} \mu y^{[m-1]},$$

hence, $x \ge (\frac{\mu}{\lambda})^{\frac{1}{m-1}} t_0 y$, consequently, $\mu \le \lambda$.
Next we assume $\mathcal{A}y^{m-1} \le \mu y^{[m-1]}$. After the remark of section 2, we have $y \in \mathrm{int}\, P$, and if we interchange the roles of $x$ and $y$ in the previous paragraph, then we have $\lambda \le \mu$. Our assertion now follows. □

We now return to the proof of Theorem 4.2:

*Proof.* After (4.3), it remains to show $r_* \le \lambda_0 \le r^*$. By the definition of $\mu_*(x)$, we have

$$r_* = \mu_*(x_*) = \mathrm{Min}_{x_i>0} \frac{(\mathcal{A}x_*^{m-1})_i}{(x_*)_i^{m-1}}.$$

This means

$$\mathcal{A}x_*^{m-1} \ge r_* x_*^{[m-1]}.$$

Likewise,

$$\mathcal{A}x^{*\,m-1} \le r^* x^{*\,[m-1]}.$$

Our desired inequality follows from Lemma 4.3. □

Since $\mu_*$ is continuous on $\Delta$ and is 0-homogeneous, we have

**Corollary 4.4.**

$$\lambda_0 = \mathrm{Max}_{x\in P\setminus\{0\}} \mathrm{Min}_{x_i>0} \frac{(\mathcal{A}x^{m-1})_i}{x_i^{m-1}}.$$

We close this section by proving assertion (4) of Theorem 1.4:

*Proof.* Let $z \in \mathbb{C}^n \setminus \{0\}$ be a solution of $\mathcal{A}z^{m-1} = \lambda z^{[m-1]}$ for some $\lambda \in \mathbb{C}$. We wish to show $|\lambda| \le \lambda_0$. Let $y_i = |z_i| \quad \forall i$ and set $y = (y_1, \dots, y_n)$. Clearly, $y \in P \setminus \{0\}$. One has

$$|(\mathcal{A}z^{m-1})_i| = |\sum_{i_2,\dots,i_m=1}^{n} a_{ii_2\cdots i_m} z_{i_2}\cdots z_{i_m}| \le \sum_{i_2,\dots,i_m=1}^{n} a_{ii_2\cdots i_m} y_{i_2}\cdots y_{i_m} = (\mathcal{A}y^{m-1})_i.$$

This shows

$$|\lambda| y_i^{m-1} = |\lambda||z_i|^{m-1} = |(\mathcal{A}z^{m-1})_i| \le (\mathcal{A}y^{m-1})_i \quad \forall i.$$

Applying Corollary 4.4, we have

$$|\lambda| \le \mathrm{Min}_{y_i>0} \frac{(\mathcal{A}y^{m-1})_i}{y_i^{m-1}} \le \mathrm{Max}_{x\in P\setminus\theta} \mathrm{Min}_{x_i>0} \frac{(\mathcal{A}x^{m-1})_i}{x_i^{m-1}} = \lambda_0.$$

□

## 5. Some Extensions

There are various ways in defining eigenvalues for tensors, e.g., there are $H$ eigenvalue, $Z$ eigenvalue, $D$ eigenvalue etc. see [7], [8],[9],[4]. They are unified in [1]. In this section, we extend the above results to more general eigenvalue problems for tensors. Let $\mathcal{A}$ and $\mathcal{B}$ be two $m$ order $n$ dimensional real tensors. Assume both $\mathcal{A}x^{m-1}$ and $\mathcal{B}x^{m-1}$ are not identically zero. We say $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is an eigen-pair (or eigenvalue and eigenvector) of $\mathcal{A}$ relative to $\mathcal{B}$, if the $n$-system of equations

$$(\mathcal{A} - \lambda\mathcal{B})x^{m-1} = 0 \tag{5.1}$$

possesses a solution.

The problem (1.5), which is called the $H$ eigenvalue problem is the case, where $\mathcal{B} = (\delta_{i_1 i_2 \dots i_m})$, the unit tensor.

We next introduce a few more conditions on nonnegative tensors.

**Definition 5.1.** (Quasi-diagonal) A tensor $\mathcal{C}$ of order $m$ dimension $n$ is said to be quasi-diagonal, if for all nonempty proper index subset $I \subset \{1,\dots,n\}$, $c_{i_1,i_2,\dots i_m} = 0$ for $i_1 \notin I$ and $i_2,\dots,i_m \in I$.

**Example 5.2.** For $m = 2$, $\mathcal{C}$ is quasi-diagonal if and only if it is a diagonal matrix.

**Example 5.3.** If $\mathcal{C} = (\delta_{i_1\dots i_m})$, where $\delta_{i_1\cdots i_m}$ is the Kronecker delta, then $\mathcal{C}$ is quasi-diagonal.

**Lemma 5.4.** If a nonnegative tensor $\mathcal{C}$ of order $m$ dimension $n$ is quasi-diagonal, then there exists $M > 0$ such that for all nonempty proper index subset $I \subset \{1,\dots,n\}$, one has $\mathcal{C}e^I \le Me^I$, where $e^I = (e^I_1,\dots,e^I_n)$ with

$$e^I_i = \begin{cases} 1, & i \in I \\ 0, & i \notin I. \end{cases}$$

*Proof.* Let $M = \sum_{i_1,\dots,i_m=1}^n c_{i_1\cdots i_m}$. We verify $(\mathcal{C}e^I)_i = 0 \quad \forall i \notin I$ by computing

$$(\mathcal{C}(e^I)^{m-1})_i = \sum_{i_2,\dots,i_m=1}^{n} c_{ii_2\cdots i_m} e^I_{i_2}\cdots e^I_{i_m} = \sum_{i_2,\dots,i_m\in I} c_{ii_2\cdots i_m} = 0, \quad \forall i \notin I,$$

provided by that $\mathcal{C}$ is quasi-diagonal. □

**Definition 5.5.** (Condition (E)) A nonnegative tensor $\mathcal{C}$ of order $m$ dimension $n$ is said to satisfy Condition (E), if there exists a homeomorphism $\tilde{C}: \mathbb{R}^n \to \mathbb{R}^n$ such that (1) $\tilde{C}|_P = C|_P$, and (2) $\forall x, y \in P, x \le y$ implies $\tilde{C}^{-1}x \le \tilde{C}^{-1}y$.

For $\mathcal{C} = (\delta_{i_1,\dots,i_m})$, $\tilde{C}$ is the identity operator, so Condition (E) is satisfied.

**Example 5.6.** Let $m$ be even, and $D$ be a positive definite matrix. If $\mathcal{C}$ is a m order n dimensional tensor satisfying:

$$\mathcal{C}x^{m-1} = Dx(Dx,x)^{\frac{m}{2}-1},$$

then $\mathcal{C}$ satisfies (1) in the Condition (E). Indeed,

$$\tilde{C}^{-1}y = D^{-1}y(y,D^{-1}y)^{-\frac{m-2}{2(m-1)}}.$$

**Example 5.7.** Let us consider the following example: let $C_k : P \to P$ be the nonlinear operator:

$$C_k x = \begin{cases} x^{[2k-1]}|x|^{2(r-k)}, & m = 2r \\ x^{[2k]}|x|^{2(r-k)} & m = 2r+1, \end{cases}$$

where $1 \le k \le r$. And let $\mathcal{C}_k = (c_{i_1,\dots,i_m})$ be an m order n dimensional nonnegative tensor corresponding to $C_k$, for example,

$$\sum c_{ii_2\dots i_m} x_{i_2}\dots x_{i_m} = \begin{cases} x_i^{2k-1}(x_1^2+\dots+x_n^2)^{r-k} & m = 2r, \\ x_i^{2k}(x_1^2+\dots+x_n^2)^{r-k} & m = 2r+1 \end{cases}$$

$$= \sum_{|\alpha|=r-k} \frac{(r-k)!}{\alpha_1!\dots\alpha_n!} x_1^{2\alpha_1}\dots x_n^{2\alpha_n} \begin{cases} x_i^{2k-1}, & m = 2r, \\ x_i^{2k} & m = 2r+1. \end{cases} \tag{5.2}$$

The left hand side equals to

$$\Sigma_{|\beta|=m-1}\Sigma_{(i_2,\dots,i_m)\sim(1^{\beta_1},\dots,n^{\beta_n})}c_{ii_2\dots i_m}x_1^{\beta_1}\dots x_n^{\beta_n},$$

where $(1^{\beta_1},....,n^{\beta_n})$ means $j$ is repeated for $\beta_j$ times, $\forall 1\le j\le n$, and $(i_2,....,i_m)\sim(i'_2,....,i'_m)$ means there exists a $\pi\in S_{m-1}$, the $m-1$ permutative group, such that $\pi(i_2,....,i_m)=(i'_2,....,i'_m)$.
(5.2) implies that

$$\beta_j=2\alpha_j+\delta_{ij}\begin{cases}2k-1 & m=2r\\ 2k & m=2r+1.\end{cases}$$

Therefore there exists a representation $\mathcal{C}_k$ of $C_k$ such that $c_{i,i_2,....,i_m}\neq 0$ only if $\exists l\ge 2$ such that $i_l=i$. Consequently, $\mathcal{C}_k$ is quasi-diagonal.
Also, $\mathcal{C}_k$ satisfies Condition (E). In fact, define

$$\tilde{C}_k=\begin{cases}x^{[2k-1]}|x|^{2(r-k)} & m=2r\\ x^{[2k]}|x|^{2(r-k)}Sgn(x) & m=2r+1,\end{cases}$$

where we use the notation: $x^{[\alpha]}Sgn(x)=(x_1^\alpha sgn(x_1),....,x_n^\alpha sgn(x_n))$. Then

$$\tilde{C}_k^{-1}=\begin{cases}y^{[\frac{1}{2k-1}]}(\Sigma_{i=1}^n|y_i|^{\frac{2}{2k-1}})^{-\frac{r-k}{m-1}} & m=2r\\ |y|^{[\frac{1}{2k}]}Sgn(y)(\Sigma_{i=1}^n|y_i|^{\frac{1}{k}})^{-\frac{r-k}{m}} & m=2r+1.\end{cases}\tag{5.3}$$

Obviously, $\tilde{C}_k$ satisfies (1) and (2) in the definition of Condition (E).

**Remark:** For $\mathcal{B}=\mathcal{C}_1$ the problem (5.1) corresponds to $Z$ eigenvalue, and when $m$ is even for $\mathcal{B}=C_{\frac{m}{2}}$ it corresponds to $H$ eigenvalue.
We have the following general result:

**Theorem 5.8.** *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are nonnegative tensors, and that $\mathcal{B}$ satisfies (1) in Condition (E), then there exist $\lambda_0\ge 0$ and a nonnegative vector $x_0\neq 0$, such that*

$$\mathcal{A}x_0^{m-1}=\lambda_0\mathcal{B}x_0^{m-1}.\tag{5.4}$$

*If further, we assume that $\mathcal{A}$ is irreducible and $\mathcal{B}$ satisfies Condition (E), and is quasi diagonal, then $x_0\in\operatorname{int}P,\lambda_0>0$ and is the unique eigenvalue with nonnegative eigenvectors. In particular, for $\mathcal{B}=\mathcal{C}_k$, the nonnegative eigenvector is unique up to a multiplicative constant.*

*Proof.* We are satisfied only to sketch the proof, because it is parallel to that in section 2. Let $A,B$ be the nonlinear operators corresponding to $\mathcal{A},\mathcal{B}$ respectively. For the existence part, we define

$$F(x)_i=\frac{(\tilde{B}^{-1}Ax))_i}{\Sigma_{j=1}^n(\tilde{B}^{-1}Ax)_j},$$

and

$$\lambda_0=(\Sigma_{j=1}^n(\tilde{B}^{-1}Ax)_j)^{m-1}$$

in replacing of (2.3) and (2.4). The after argument is the same with the counter part in section 2.

Next we follow step 1° in the proof of theorem 2 to prove $x_0 \in \text{int}\, P$ by contradiction. Suppose not, then there exists a maximal proper index subset $I$ such that $x_0 \in F_I$. From the equation:

$$\mathcal{A}x_0^{m-1} = \lambda_0 \mathcal{B}x_0^{m-1},$$

and that $\mathcal{B}$ is quasi diagonal, it follows $Bx_0 \in F_I$ and then $Ax_0 \in F_I$. The following arguments are the same.

In step 3°, (2.5) is replaced by

$$\lambda Bx = Ax \geq \mu \delta_x(-y)^{m-1} By.$$

Since $\tilde{B}^{-1}$ is order preserving in $P$, and is positively $\frac{1}{m-1}$ homogeneous, we have

$$\lambda^{\frac{1}{m-1}} x \geq \mu^{\frac{1}{m-1}} \delta_x(-y) y.$$

Therefore $\mu \leq \lambda$. Again the rest part is the same.

As to the uniqueness of the positive eigenvector (up to a multiplicative constant), we reduce the problem by changing variables. For $x \neq 0$, let

$$\xi = \begin{cases} x|x|^{-\frac{2(r-k)}{m-\alpha-1}}, & m = 2r \\ x|x|^{-\frac{2(r-k)}{m-\alpha-2}}, & m = 2r+1, \end{cases}$$

where $\alpha = 2k$ when $m$ is even, and $2k+1$ when $m$ is odd. The problem is then reduced to:

$$A\xi = \lambda_0 \xi^{[\alpha]}.$$

We shall prove that the nonnegative eigenvector $x_0$ is unique. It is proved by contradiction. Suppose not, there exist $x, y \in P\backslash\{0\}$ satisfying $Ax = \lambda_0 x$ and $Ay = \lambda_0 y$. Let $\xi, \eta$ are the images of $x, y$ under the above transformation. Then by the argument in step 4° of the proof in Theorem 1.4 there exists $t > 0$ such that $\xi = t\eta$. This implies

$$x = t(\frac{|x|}{|y|})^{\frac{2(r-k)}{m-\alpha-1}} y,$$

where $\alpha = 2k$ or $2k+1$ if $m = 2r$ or $2r+1$ resp.

Lastly, the minimax theorem in §4 is also extended:

$$Min_{x \in \text{int}\, P} Max_{x_i > 0} \frac{(\mathcal{A}x^{m-1})_i}{(\mathcal{C}_k x^{m-1})_i} = \lambda_0 = Max_{x \in \text{int}\, P} Min_{x_i > 0} \frac{(\mathcal{A}x^{m-1})_i}{(\mathcal{C}_k x^{m-1})_i}.$$

The following steps follow the last paragraph of §4. □

**Corollary 5.9.** Theorem 1.3 holds for D-eigenvalue problem. Theorem 1.4 holds for H eigenvalue problem and Z eigenvalue problem.

More generally, For $Bx = x^{[k]}\varphi(x)$, where $1 \leq k \leq [\frac{m}{2}]$ and $\varphi(x)$ is a positively $m-k-1$ homogeneous positive polynomial, Perron Frobenius theorem also holds.

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LMAM, School of Math. Sci., Peking Univ., Beijing 100871, China, kcchang@math.pku.edu.cn

Dept. Math & Stats., Murray State Univ., Murray, KY 42071, USA, kelly.pearsong@murraystate.edu

Dept. Math & Stats., Murray State Univ., Murray, KY 42071, USA, tan.zhang@murraystate.edu