On a Kind of Integrals of Empirical Processes Concerning Insurance Risk *

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Abstract: In this paper we prove the strong consistence and the central limit theorems for empirical right tail deviations.

Keywords: Insurance premium, Right tail deviation, Skorokhod construction, Empirical processes.

1 Introduction and main results

In actuarial science, an insurance risk X is usually defined as non-negative random variable (r.v.) and its premium refers to a functional $H(X) : X \to [0, \infty)$. Let F be the distribution function (d.f.) of X and denote $S = 1 - F$. Wang [1],[2] defined the so called PH-transform premium as

$$H(X) = H_\alpha(X) = \int_0^\infty S^\alpha(x)dx, \tag{1.1}$$

where $\alpha \in (0, 1)$ is constant. Suppose that the expectation EX of X exists. Then we have

$$EX = \int_0^\infty S(x)dx.$$

Another important quantity is the risk loading $D(X) := H(X) - EX$. If $H_\alpha(X)$ is defined as (1.1) with $\alpha = 1/2$, then

$$D(X) = \int_0^\infty \sqrt{S(x)}dx - \int_0^\infty S(x)dx$$

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is named as the right tail deviation by Wang [3]. This was generalized by Li [4] who also called

$$D_\alpha(X) := H_\alpha(X) - E(X) = \int_0^\infty S^\alpha(x)dx - \int_0^\infty S(t)dt \tag{1.2}$$

as the right tail deviation. It follows from Wang, Young and Panjer [5] that under five reasonable axioms for insurance principle H, there exists $\alpha \geq 0$ such that (1.1) holds and under a natural restriction $H(X) \geq E(X)$, the α has to be in the interval $(0, 1]$. From now on, we shall assume $\alpha \in (0, 1)$ since the case $\alpha = 1$ is trivial.

Let $\{X_1, X_2, \cdots\}$ be independent and identically distributed (i.i.d.) random variables defined on probability space $\{\Omega, \mathcal{F}, P\}$ with the common d.f. F, i.e. $\{X_1, \cdots, X_n\}$ be sample of size n from the population F for $n = 1, 2, \cdots$. In practice, we do not know the F exactly. Hence Wang [3] suggested replacing F by its empirical d.f.

$$F_n(x) := \frac{1}{n}\sum_{i=1}^n I_{\{X_i \leq x\}}, x \in [0, \infty).$$

Therefore the estimators of $H_\alpha(X)$ and $D_\alpha(X)$ are

$$\begin{aligned} H_{n,\alpha}(X) &:= \int_0^\infty [1 - F_n(x)]^\alpha dx; \\ D_{n,\alpha}(X) &:= \int_0^\infty [1 - F_n(x)]^\alpha dx - \int_0^\infty [1 - F_n(x)]dx. \end{aligned}$$

Thus problems arise as follows: how are $H_{n,\alpha}(X)$ and $D_{n,\alpha}(X)$ respectively close to $H_\alpha(X)$ and $D_\alpha(X)$?For the weak consistency of $D_{n,\alpha}(X)$, one may see Li [4]. This paper is to devote to discussing strong consistency and the asymptotic normality for those two estimators.

As for the strong consistency, our result may be stated as follows.

Theorem 1 If there exists $\delta > 0$ such that

$$EX_1^{\alpha^{-1}+\delta} < \infty, \tag{1.3}$$

then

$$\lim_{n\to\infty} H_{n,\alpha}(X) = H_\alpha(X) \quad a.s.; \tag{1.4}$$

$$\lim_{n\to\infty} D_{n,\alpha}(X) = D_\alpha(X) \quad a.s.. \tag{1.5}$$

Denote the left continuous inversion of F by

$$F^{\leftarrow}(t) := \inf\{x \in R : F(x) \geq t\}, \forall t \in (0, 1).$$

Our second result is about the asymptotic normality of $H_{n,\alpha}(X)$ and $D_{n,\alpha}(X)$.

Theorem 2 If there exists $\delta \in (1/2, 1)$ such that

$$\int_{(0,1)} (1 - t)^{\alpha-\delta} dF^{\leftarrow}(t) < \infty, \tag{1.6}$$

then as $n \to \infty$,

$$\sqrt{n}[H_{n,\alpha}(X) - H_\alpha(X)] \quad \xrightarrow{d} \quad N(0, \alpha^2\sigma^2_{H,\alpha}); \tag{1.7}$$

$$\sqrt{n}[D_{n,\alpha}(X) - D_\alpha(X)] \quad \xrightarrow{d} \quad N(0, \sigma^2_{D,\alpha}), \tag{1.8}$$

where

$$\sigma^2_{H,\alpha} \quad := \quad \int\int_{s,t\in(0,1)} \frac{s \wedge t - st}{[(1-s)(1-t)]^{1-\alpha}} dF^{\leftarrow}(s)dF^{\leftarrow}(t); \tag{1.9}$$

$$\sigma^2_{D,\alpha} \quad := \quad \int\int_{s,t\in(0,1)} (s \wedge t - st)J(s)J(t)dF^{\leftarrow}(s)dF^{\leftarrow}(t) \tag{1.10}$$

and $J(t) = 1 - \alpha(1-t)^{-(1-\alpha)}$.

The following Corollary is easily obtained.

Corollary of Theorem 2 If

$$\int_{(0,1)} (1-t)^{1-\alpha} dF^{\leftarrow}(t) < \infty$$

holds for $\alpha \in (3/4, 1)$, then as $n \to \infty$, (1.7) and (1.8) hold.

In section 2, we will give the proof of Theorem 1. In section 3, some lemmas are proved on the Skorokhod construction, then in section 4 the proof of Theorem 2 is given.

2 Proof of Theorem 1

We shall denote the value of X at $\omega \in \Omega$ by $X_i(\omega)$ for $i = 1, 2, \cdots$ and the value of $F_n(x)$ at $\omega \in \Omega$ by $F_n(x, \omega)$ for all $x \in [0, \infty)$.

Let

$$\Omega_1 \quad = \quad \{\omega \in \Omega : \lim_{n\to\infty} \sup_t \mid F_n(t,\omega) - F(t) \mid = 0\},$$

$$\Omega_2 \quad = \quad \{\omega \in \Omega : \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n X_i^{\alpha^{-1}+\delta}(\omega) = EX_1^{\alpha^{-1}+\delta}\}$$

and $\Omega_0 = \Omega_1 \cap \Omega_2$. Then we have $P(\Omega_1) = 1$ by the Glivenko-Cantelli theorem ([6], page 52) and $P(\Omega_2) = 1$ by the strong laws of large numbers ([6], page 51). Therefore $P(\Omega_0) = 1$ holds.

Fix $\omega \in \Omega_0$. For any $T > 0$, we have

$$\begin{aligned}
& \int_T^\infty \Big[1 - F_n(t,\omega)\Big]^\alpha dt = \int_T^\infty \Big[\int_t^\infty F_n(dx,\omega)\Big]^\alpha dt \qquad (2.1) \\
\leq & \int_T^\infty \Big[\int_t^\infty \frac{x^{\alpha^{-1}+\delta}}{t^{\alpha^{-1}+\delta}} F_n(dx,\omega)\Big]^\alpha dt \\
= & \int_T^\infty \frac{1}{t^{1+\alpha\delta}} \Big[\int_t^\infty x^{\alpha^{-1}+\delta} F_n(dx,\omega)\Big]^\alpha dt \\
\leq & \int_T^\infty \frac{1}{t^{1+\alpha\delta}} \Big[\int_0^\infty x^{\alpha^{-1}+\delta} F_n(dx,\omega)\Big]^\alpha dt \\
= & \frac{T^{-\alpha\delta}}{\alpha\delta} \{\frac{\sum_{i=1}^n X_i^{\alpha^{-1}+\delta}(\omega)}{n}\}^\alpha \\
\to & \frac{T^{-\alpha\delta}}{\alpha\delta} (EX_1^{\alpha^{-1}+\delta})^\alpha
\end{aligned}$$

as $n \to \infty$. It is easily seen that (1.3) implies $\int_0^\infty [1 - F(t)]^\alpha dt < \infty$. Therefore we also have

$$\lim_{T\to\infty} \int_T^\infty [1 - F(t)]^\alpha dt = 0. \qquad (2.2)$$

In the following inequality

$$\begin{aligned}
& \left|\int_0^\infty [1 - F_n(t,\omega)]^\alpha dt - \int_0^\infty [1 - F(t)]^\alpha dt\right| \\
\leq & \left|\int_T^\infty [1 - F_n(t,\omega)]^\alpha dt\right| + \left|\int_T^\infty [1 - F(t)]^\alpha dt\right| \\
& + \left|\int_0^T [1 - F_n(t,\omega)]^\alpha dt - \int_0^T [1 - F(t)]^\alpha dt\right|,
\end{aligned}$$

let $n \to \infty$ and $T \to \infty$ in turn. Then (1.4) follows from the Glivenko-Cantelli theorem, the dominated convergence theorem, limit (2.1) and limit (2.2). Note that (1.3) implies $E|X_1| < \infty$ and therefore by the strong laws of large numbers,

$$\int_0^\infty [1 - F_n(t)]dt = \frac{1}{n}\sum_{i=1}^n X_i \to EX_1 \quad a.s. \quad as \quad n \to \infty.$$

The limit (1.5) is also obtained by combining the above with (1.4).

3 Some lemmas on the Skorokhod construction

Let D denote the space consisting of all functions on $[0,1]$, that are right continuous and possess left-hand limits at each point. For any $f, g \in D$, define the uniform metric

$$\|f-g\| = \sup_{t\in[0,1]} |f(t)-g(t)|.$$

Then the Skorokhod construction ([7], page 93) may be stated as follows: *There exists a probability space* $(\Omega, \mathcal{F}, P)$ *on which a triangular array of row-independent uniform* $[0,1]$ *r.v.'s* $\{\xi_{n,1}, \xi_{n,2}, \cdots, \xi_{n,n}, n = 1, 2, \cdots\}$ *and a Brown bridge* U *are defined such that*

$$\lim_{n\to\infty} \|U_n - U\| = 0 \qquad a.s.$$

where U_n *denotes the empirical processes determined by* $\xi_{n,1}, \xi_{n,2}, \cdots, \xi_{n,n}$*, that is,*

$$U_n(t) = \sqrt{n}[\Gamma_n(t) - t]$$

with $\Gamma_n(t) = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} I_{\{\xi_{n,i}\le t\}}, \forall t \in [0,1]$.

Moreover, for the Skorokhod construction, we have the following property ([7], page 140): *if* q *is a nondecreasing function on* $[0,1/2]$*, symmetric about* $1/2$ *and satisfying* $\displaystyle\int_0^1 q^{-2}(t)dt < \infty$*, then*

$$\left\|\frac{U_n - U}{q}\right\| \xrightarrow{P} 0, \quad n \longrightarrow \infty. \tag{3.1}$$

In the following, notation $\eta \sim N(\mu, \sigma^2)$ will be used to denote a normal r.v. η with mean μ and variance σ^2 if $\sigma^2 > 0$ or a degenerate r.v. at μ if $\sigma^2 = 0$. Our first lemma in this section is about the Brown bridge U.

Lemma 1 If

$$\int_{(0,1)} (1-t)^{\alpha-1/2} dF^{\leftarrow}(t) < \infty, \tag{3.2}$$

then $\sigma^2_{H,\alpha}$ and $\sigma^2_{D,\alpha}$ defined in (1.9) and (1.10) are finite. Moreover, it holds that

$$\psi(U) := \int_{(0,1)} \frac{U(t)}{(1-t)^{1-\alpha}} dF^{\leftarrow}(t) \sim N(0, \sigma^2_{H,\alpha}); \tag{3.3}$$

$$\phi(U) := \int_{(0,1)} U(t)J(t)dF^{\leftarrow}(t) \sim N(0, \sigma^2_{D,\alpha}). \tag{3.4}$$

Proof We shall show $\sigma^2_{H,\alpha}$ is finite and (3.3) holds. The proofs of the finiteness of $\sigma^2_{D,\alpha}$ and (3.4) are similar.

It follows from the Fubini theorem, the Schwarz inequality and (3.2) that

$$\begin{aligned} & E\int_{(0,1)} |U(t)|(1-t)^{\alpha-1}dF^{\leftarrow}(t) \\ = & \int_{(0,1)} [E|U(t)|](1-t)^{\alpha-1}dF^{\leftarrow}(t) \\ \leq & \int_{(0,1)} \sqrt{EU^2(t)}(1-t)^{\alpha-1}dF^{\leftarrow}(t) \\ = & \int_{(0,1)} \sqrt{t(1-t)}(1-t)^{\alpha-1}dF^{\leftarrow}(t) \\ \leq & \int_{(0,1)} (1-t)^{\alpha-1/2}dF^{\leftarrow}(t) < \infty. \end{aligned}$$

Therefore we have

$$|\psi(U)| \leq \int_{(0,1)} |U(t)|(1-t)^{\alpha-1}dF^{\leftarrow}(t) < \infty \quad a.s.,$$

i.e. ψ determines a random variable. By using Fubini theorem, the Schwarz inequality and (3.2) again, we get

$$\begin{aligned} E\psi^2(U) &= E\Big\{\int_{(0,1)} U(t)(1-t)^{\alpha-1}dF^{\leftarrow}(t)\Big\}^2 \qquad (3.5) \\ &= E\Big\{\int_{(0,1)} \frac{U(t)}{\sqrt{1-t}}\cdot(1-t)^{\alpha-1/2}dF^{\leftarrow}(t)\Big\}^2 \\ &\leq E\Big\{\int_{(0,1)} (\frac{U^2(t)}{1-t})\cdot(1-t)^{\alpha-1/2}dF^{\leftarrow}(t)\Big\}\cdot\Big\{\int_{(0,1)} (1-t)^{\alpha-1/2}dF^{\leftarrow}(t)\Big\} \\ &\leq \Big\{\int_{(0,1)} (1-t)^{\alpha-1/2}dF^{\leftarrow}(t)\Big\}^2 < \infty. \end{aligned}$$

Hence the variance of the r.v. $\psi(U)$ exists. For any $0 < a < b < 1$ and $n = 1, 2, \cdots$, let $x_{n,i} = a + i(b-a)/n$ for $i = 1, \cdots, n$, then we have from the definition of Riemann-Stieltjes integral that

$$\psi(U) = \lim_{a\downarrow 0, b\uparrow 1}\lim_{n\to\infty}\sum_{i=1}^{n}\frac{U(x_{n,i})}{(1-x_{n,i})^{1-\alpha}}\left[F^{\leftarrow}(x_{n,i}) - F^{\leftarrow}(x_{n,i-1})\right]. \qquad (3.6)$$

Note that if a limit r.v. of sequence consisting of normal r.v.'s with mean 0 and degenerate r.v.'s at 0 has finite variance, then it has to be normal with mean 0 or degenerate at 0. We obtain $\psi \sim N(0, E\psi^2(U))$ from (3.5) and (3.6), where

$$\begin{aligned} E\psi^2(U) &= E\Big\{\int_{(0,1)} U(t)(1-t)^{\alpha-1} dF^{\leftarrow}(t)\Big\}^2 \\ &= E\int\int_{s,t\in(0,1)} U(s)U(t)[(1-s)(1-t)]^{\alpha-1} dF^{\leftarrow}(s)dF^{\leftarrow}(t) \\ &= \int\int_{s,t\in(0,1)} [EU(s)U(t)][(1-s)(1-t)]^{\alpha-1} dF^{\leftarrow}(s)dF^{\leftarrow}(t) \\ &= \int\int_{s,t\in(0,1)} \frac{s\wedge t - st}{[(1-s)(1-t)]^{1-\alpha}} dF^{\leftarrow}(s)dF^{\leftarrow}(t) = \sigma^2_{H,\alpha}, \end{aligned}$$

completing the proof of the lemma.

Lemma 2 For the Skorokhod construction, if there exists $\delta \in (1/2, 1)$ such that (1.6) holds, then as $n \to \infty$,

$$\int_{(0,1)} \frac{U_n(t)}{(1-t)^{1-\alpha}} dF^{\leftarrow}(t) \xrightarrow{P} \int_{(0,1)} \frac{U(t)}{(1-t)^{1-\alpha}} dF^{\leftarrow}(t); \tag{3.7}$$

$$\int_{(0,1)} U_n(t)J(t)dF^{\leftarrow}(t) \xrightarrow{P} \int_{(0,1)} U(t)J(t)dF^{\leftarrow}(t). \tag{3.8}$$

Proof We shall show (3.7) only. The proof of (3.8) is similar. Let $q(t) = [t(1-t)]^{1-\delta}$. Then (3.1) holds since

$$\int_0^1 q^{-2}(t)dt = \int_0^1 t^{(2\delta-1)-1}(1-t)^{(2\delta-1)-1}dt < \infty.$$

Therefore denoting $I(t) = t, \forall t \in (0,1)$, we get

$$\begin{aligned} & \Big|\int_{(0,1)} \frac{U_n(t)}{(1-t)^{1-\alpha}} dF^{\leftarrow}(t) - \int_{(0,1)} \frac{U(t)}{(1-t)^{1-\alpha}} dF^{\leftarrow}(t)\Big| \\ \leq & \left\|\frac{U_n - U}{(1-I)^{1-\delta}}\right\| \int_{(0,1)} (1-t)^{\alpha-\delta} dF^{\leftarrow}(t) \\ \leq & \left\|\frac{U_n - U}{q}\right\| \int_{(0,1)} (1-t)^{\alpha-\delta} dF^{\leftarrow}(t) \xrightarrow{P} 0 \end{aligned}$$

as $n \to \infty$, so that (3.7) holds.

For the Skorokhod construction, the order statistics of $\xi_{n,1}, \cdots, \xi_{n,n}$ is denoted by $\xi_{n:1} \leq \cdots \leq \xi_{n:n}$ for $n = 1, 2, \cdots$ and we define

$$\Delta_n^\star := \int_{[\xi_{n:n},1)} (1-t)^\alpha dF^{\leftarrow}(t);$$

$$\delta_n^\star := \frac{1}{\sqrt{n}} \int_{(0,\xi_{n:n})} \frac{U_n^2(t)}{(1-t)^{1-\alpha}\{(1-t)\wedge[1-\Gamma_n(t)]\}} dF^{\leftarrow}(t).$$

Lemma 3 For Skorokhod construction, if (1.6) holds, then as $n \to \infty$,

$$\sqrt{n}\Delta_n^\star \xrightarrow{P} 0; \tag{3.9}$$

$$\delta_n^\star \xrightarrow{P} 0. \tag{3.10}$$

Proof Note that

$$P\{n(1-\xi_{n:n}) \le x\} = P\{n\xi_{n:1} \le x\} = 1 - (1-\frac{x}{n})^n \to 1 - e^{-x}, \forall x \in [0,\infty), \tag{3.11}$$

i.e. $\{n(1-\xi_{n:n})\}$ converges in distribution to a standard exponential *r.v.*. Since $\xi_{n:n} \xrightarrow{P} 1$, (1.6) implies

$$\int_{[\xi_{n:n},1)} (1-t)^{\alpha-1/2} dF^{\leftarrow}(t) \xrightarrow{P} 0.$$

Then we get as $n \to \infty$,

$$\begin{aligned} \sqrt{n}\Delta_n^\star &= \sqrt{n}\int_{[\xi_{n:n},1)} (1-t)^{1/2}(1-t)^{\alpha-1/2} dF^{\leftarrow}(t) \\ &\le \sqrt{n(1-\xi_{n:n})}\int_{[\xi_{n:n},1)} (1-t)^{\alpha-1/2} dF^{\leftarrow}(t) \xrightarrow{P} 0. \end{aligned}$$

This proves (3.9).

It is easily seen that

$$\begin{aligned} \delta_n^\star &= \frac{1}{\sqrt{n}}\int_{(0,\xi_{n:n})} \frac{U_n^2(t)}{(1-t)^{2-\alpha}}\left\{1 \vee \frac{(1-t)}{1-\Gamma_n(t)}\right\} dF^{\leftarrow}(t) \\ &\le \left\{1 + \sup_{t\in(0,\xi_{n:n})}\left|\frac{1-t}{1-\Gamma_n(t)}\right|\right\} \cdot \frac{1}{\sqrt{n}}\int_{(0,\xi_{n:n})} \frac{U_n^2(t)}{(1-t)^{2-\alpha}} dF^{\leftarrow}(t) \\ &= \left\{1 + \sup_{t\in[\hat{\xi}_{n:1},1)}\left|\frac{t}{\hat{\Gamma}_n(t)}\right|\right\} \cdot \frac{1}{\sqrt{n}}\int_{(0,\xi_{n:n})} \frac{U_n^2(t)}{(1-t)^{2-\alpha}} dF^{\leftarrow}(t), \end{aligned}$$

where $\hat{\xi}_{n:i} = 1 - \xi_{n:n-i+1}$ and $\hat{\Gamma}_n$ is the empirical d.f. determined by $\hat{\xi}_{n:i}, i = 1, 2, \cdots, n$. By the inequality in [7] (page 451), we have

$$P\left\{\sup_{t\in[\hat{\xi}_{n:1},1)}\left|\frac{t}{\hat{\Gamma}_n(t)}\right| \ge \lambda\right\} \le \lambda e^{-\lambda+1}, \forall\lambda > 0,$$

and therefore for all $\lambda, \eta > 0$,

$$P\{\delta_n^\star \ge \eta\} \le P\left\{\frac{\lambda+1}{\sqrt{n}}\int_{(0,\xi_{n:n})} \frac{U_n^2(t)}{(1-t)^{2-\alpha}} dF^{\leftarrow}(t) \ge \eta\right\} + \lambda e^{-\lambda+1}. \tag{3.12}$$

Since (3.11) and

$$\begin{aligned} & E\int_{(0,1)}\frac{U_n^2(t)}{1-t}\cdot(1-t)^{\alpha-\delta}dF^{\leftarrow}(t) \\ = & \int_{(0,1)}\frac{EU_n^2(t)}{1-t}\cdot(1-t)^{\alpha-\delta}dF^{\leftarrow}(t) \\ \leq & \int_{(0,1)}(1-t)^{\alpha-\delta}dF^{\leftarrow}(t)<\infty \end{aligned} \tag{3.13}$$

holds, we get

$$\begin{aligned} & \frac{1}{\sqrt{n}}\int_{(0,\xi_{n:n})}\frac{U_n^2(t)}{(1-t)^{2-\alpha}}dF^{\leftarrow}(t)=\frac{1}{\sqrt{n}}\int_{(0,\xi_{n:n})}\frac{(1-t)^{\alpha-\delta}U_n^2(t)}{(1-t)^{1+(1-\delta)}}dF^{\leftarrow}(t) \\ \leq & \ n^{-(\delta-1/2)}\cdot[n(1-\xi_{n:n})]^{-(1-\delta)}\int_{(0,1)}\frac{U_n^2(t)}{1-t}\cdot(1-t)^{\alpha-\delta}dF^{\leftarrow}(t)\xrightarrow{P}0. \end{aligned}$$

Hence (3.10) follows by letting $n\to\infty$ and $\lambda\to\infty$ in (3.12), completing the proof of the lemma.

Lemma 4 For the Skorokhod construction, if (1.6) holds, then as $n\to\infty$,

$$\int_{[\xi_{n:n},1)}\frac{|U_n(t)|}{(1-t)^{1-\alpha}}dF^{\leftarrow}(t)\xrightarrow{P}0. \tag{3.14}$$

Proof The limit (3.14) follows from (3.13), the fact that $\xi_{n:n}\xrightarrow{P}1$ and the following inequality

$$\begin{aligned} & \int_{[\xi_{n:n},1)}\frac{|U_n(t)|}{(1-t)^{1-\alpha}}dF^{\leftarrow}(t)=\int_{[\xi_{n:n},1)}\frac{|U_n(t)|}{(1-t)^{1-\delta}}\cdot(1-t)^{\alpha-\delta}dF^{\leftarrow}(t) \\ \leq & \ (1-\xi_{n:n})^{\delta-1/2}\int_{(0,1)}\frac{|U_n(t)|}{\sqrt{1-t}}\cdot(1-t)^{\alpha-\delta}dF^{\leftarrow}(t) \\ \leq & \ (1-\xi_{n:n})^{\delta-1/2}\left\{\int_{(0,1)}\frac{U_n^2(t)}{1-t}\cdot(1-t)^{\alpha-\delta}dF^{\leftarrow}(t)\right\}^{1/2}\cdot\left\{\int_{(0,1)}(1-t)^{\alpha-\delta}dF^{\leftarrow}(t)\right\}^{1/2}. \end{aligned}$$

4 The proof of Theorem 2

For any nonnegative Borel measurable function f defined on (0,1), a typical argument leads to

$$\int_{-\infty}^{\infty}f(F(x))dx=\int_{(0,1)}f(t)dF^{\leftarrow}(t).$$

Therefore we have

$$\begin{aligned} H_{n,\alpha}(X) - H_\alpha(X) &= \int_0^\infty [1-F_n(x)]^\alpha dx - \int_0^\infty [1-F(x)]^\alpha dx \\ &\stackrel{d}{=} \int_0^\infty [1-\Gamma_n(F(x))]^\alpha dx - \int_0^\infty [1-F(x)]^\alpha dx \\ &= \int_{(0,1)} [1-\Gamma_n(t)]^\alpha dF^{\leftarrow}(t) - \int_{(0,1)} (1-t)^\alpha dF^{\leftarrow}(t) \end{aligned}$$

with the uniform empirical d.f. Γ_n defined in the Skorokhod construction, and similarly

$$\begin{aligned} \Delta_n: &= D_{n,\alpha}(X) - D_\alpha(X) \\ &\stackrel{d}{=} \int_{(0,1)} \{[1-\Gamma_n(t)]^\alpha - (1-t)^\alpha\} dF^{\leftarrow}(t) + \frac{1}{\sqrt{n}} \int_{(0,1)} U_n(t) dF^{\leftarrow}(t). \end{aligned} \tag{4.1}$$

Proof We will prove (1.8) only, the proof of (1.7) is similar. For this, we need the following inequality

$$\frac{\alpha(y-x)}{x^{1-\alpha}}\left\{1-\frac{(1-\alpha)(y-x)}{x\wedge y}\right\} \le y^\alpha - x^\alpha \le \frac{\alpha(y-x)}{x^{1-\alpha}}, \forall x,y>0, \tag{4.2}$$

which may be obtained as follows. Since the function $f(x)=x^\alpha, x>0$ is concave, we have

$$\frac{\alpha(y-x)}{y^{1-\alpha}} \le y^\alpha - x^\alpha \le \frac{\alpha(y-x)}{x^{1-\alpha}}, \forall x,y \ge 0. \tag{4.3}$$

And one can prove that

$$\begin{aligned} \frac{y^{1-\alpha}-x^{1-\alpha}}{y^{1-\alpha}} &\le \frac{(1-\alpha)x^{-\alpha}(y-x)}{y^{1-\alpha}} \le \frac{(1-\alpha)(y-x)}{x}, \forall y>x>0, \\ \frac{y^{1-\alpha}-x^{1-\alpha}}{y^{1-\alpha}} &\ge \frac{(1-\alpha)y^{-\alpha}(y-x)}{y^{1-\alpha}} = \frac{(1-\alpha)(y-x)}{y}, \forall 0<y<x. \end{aligned}$$

Following the above two inequalities,

$$\begin{aligned} \frac{\alpha(y-x)}{y^{1-\alpha}} &= \frac{\alpha(y-x)}{x^{1-\alpha}}\left\{1-\frac{y^{1-\alpha}-x^{1-\alpha}}{y^{1-\alpha}}\right\} \\ &\ge \frac{\alpha(y-x)}{x^{1-\alpha}}\left\{1-\frac{(1-\alpha)(y-x)}{x\wedge y}\right\}, \forall x,y>0 \end{aligned}$$

holds. Combining the above inequality with (4.3), (4.2) is proved.

It follows from (4.1) that

$$\Delta_n = D_{n,\alpha}(X) - D_\alpha(X) \stackrel{d}{=} \Delta_{n,1} + \Delta_{n,2} + \Delta_{n,3}, \tag{4.4}$$

where

$$\Delta_{n,1} := \int_{(0,\xi_{n:n})} \{[1-\Gamma_n(t)]^\alpha - (1-t)^\alpha\}\, dF^{\leftarrow}(t);$$
$$\Delta_{n,2} := \frac{1}{\sqrt{n}}\int_{(0,1)} U_n(t) dF^{\leftarrow}(t);$$
$$\Delta_{n,3} := \int_{[\xi_{n:n},1)} \{[1-\Gamma_n(t)]^\alpha - (1-t)^\alpha\}\, dF^{\leftarrow}(t).$$

For $t \geq \xi_{n:n}$, $\Gamma_n(t) = 1$, so

$$\Delta_{n,3} = -\int_{[\xi_{n:n},1)} (1-t)^\alpha dF^{\leftarrow}(t) = -\Delta_n^\star.$$

Then by (3.9) we have

$$\sqrt{n}|\Delta_{n,3}| = \sqrt{n}\Delta_n^\star \xrightarrow{P} 0.$$

Therefore from (4.4) the sequences of r.v.'s $\{\sqrt{n}\Delta_n\}$ and $\{\sqrt{n}[\Delta_{n,1}+\Delta_{n,2}]\}$ have the same asymptotic d.f..

By using the inequality (4.2), it holds that

$$-\frac{\alpha[\Gamma_n(t)-t]}{(1-t)^{1-\alpha}} - \frac{\alpha(1-\alpha)[\Gamma_n(t)-t]^2}{(1-t)^{1-\alpha}\{(1-t)\wedge(1-\Gamma_n(t))\}}$$
$$\leq [1-\Gamma_n(t)]^\alpha - (1-t)^\alpha \leq -\frac{\alpha[\Gamma_n(t)-t]}{(1-t)^{1-\alpha}}, \forall t \in (0,\xi_{n:n}).$$

Hence denoting $\delta_n := \int_{(0,\xi_{n:n})} \frac{U_n(t)}{(1-t)^{(1-\alpha)}} dF^{\leftarrow}(t)$ and letting $\delta_n^\star$ as in Lemma 3, we get further

$$\begin{aligned}
-\alpha\delta_n - \alpha(1-\alpha)\delta_n^\star + \sqrt{n}\Delta_{n,2} &\leq \sqrt{n}(\Delta_{n,1}+\Delta_{n,2}) \\
&\leq -\alpha\delta_n + \sqrt{n}\Delta_{n,2} \\
&= -\alpha\int_{(0,\xi_{n:n})} \frac{U_n(t)}{(1-t)^{1-\alpha}} dF^{\leftarrow}(t) + \int_{(0,1)} U_n(t) dF^{\leftarrow}(t).
\end{aligned}$$

Combining the above with (3.10), we assert that the sequence of r.v.'s $\{\sqrt{n}[\Delta_{n,1}+\Delta_{n,2}]\}$ and the sequence of r.v.'s

$$\Big\{-\alpha\int_{(0,\xi_{n:n})} \frac{U_n(t)}{(1-t)^{1-\alpha}} dF^{\leftarrow}(t) + \int_{(0,1)} U_n(t) dF^{\leftarrow}(t)\Big\}$$

have the same asymptotic d.f..

Finally, it follows from (3.14) that the sequence

$$\Big\{-\alpha\int_{(0,\xi_{n:n})} \frac{U_n(t)}{(1-t)^{1-\alpha}} dF^{\leftarrow}(t) + \int_{(0,1)} U_n(t) dF^{\leftarrow}(t)\Big\}$$

and the sequence

$$\{Z_n := -\alpha \int_{(0,1)} \frac{U_n(t)}{(1-t)^{1-\alpha}} dF^{\leftarrow}(t) + \int_{(0,1)} U_n(t) dF^{\leftarrow}(t) = \int_{(0,1)} U_n(t) J(t) dF^{\leftarrow}(t)\}$$

have the same asymptotic d.f. . Now we have shown that the sequences $\{\sqrt{n}\Delta_n\}$ and $\{Z_n\}$ have the same limit d.f.. By Lemma 1 and Lemma 2, the limit distribution of the sequence $\{Z_n\}$ is $N(0, \sigma^2_{D,\alpha})$. Then by the definition of (4.1), the limit distribution of the sequence $\{\sqrt{n}[D_{n,\alpha}(X) - D_\alpha(X)]\}$ is $N(0, \sigma^2_{D,\alpha})$, completing the proof of (1.8).

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