Joint-life status and Gompertz's law

Jingping Yang, Shulin Zhou
Department of Financial Mathematics, Peking University, Beijing 100871

Abstract In this paper we prove that in a survivorship group, the force of mortality of the group must follow Gompertz's law provided that, for the joint-life status of every two lives, one can find a single-life status whose time-until-death's distribution equals that of the joint-life status. Therefore, the assumption that the force of mortality follows Gompertz's law is the necessary and sufficient criterion to guarantee that every joint-life status' survival pattern can be replaced by a single-life status' in the group.

Keywords Joint-life status, Gompertz's law, Force of mortality

MR(1991)Subjext Classification 90A99; 62P05

1 Introduction

In annuities and insurances, multiple life models play an important role, and the applications of multiple life actuarial models are common. For example, in estate and gift taxation the investment income from a trust can be paid to a group of heirs as long as at least one of the group survives. See [1],[2] and [3] for more examples. In this paper, we discuss the condition under which multiple life models can be replaced by single life models.

In a survivorship group, for the single life model, we use X to represent one individual's *age-at-death*. And (x) is used to denote a *life-age-x*. For the individual aged x, $T(x) = X - x$ denotes the *future lifetime* of the life (x). Thus, the random variable $T(x)$ can be interpreted as the period of survival of the status and also as the *time-until-failure* of the status. For the multiple lives (x_1), (x_2), $\ldots$, (x_m), the *joint-life status* $(x_1 x_2 \cdots x_m)$ (one multiple life model), is defined as a status that exists as long as all lives survive and fails upon the first death. The time-until-failure of the joint-life status is defined as

$$T(x_1, x_2, \cdots, x_n) = \min[T(x_1), T(x_2), \ldots, T(x_n)],$$

which is the time-until-death of the life who first dies. In this paper we restrict our attention to the two-life case, and assume they are independent.

In actuarial science and demography, one important approach to describe the survival pattern is the force of mortality. We use $\mu(t)$ to denote the group's force of mortality. It is defined mathematically as

$$\mu(t) = \frac{f_X(t)}{1 - F_X(t)}, t \in [0, \infty)$$

where $f_X(t)$ and $F_X(t)$ are respectively the density function and the distribution function of X. For each age t, it gives the value of the conditional probability density function (p.d.f.) of X at exact age t, given survival to that age. It is easy to prove that (x)' force of mortality, denoted by $\mu_x(t)$, is equal to $\mu(x+t)$. For (xy)'s force of mortality, we denote by $\mu_{xy}(t)$. In this paper we assume $\mu(t)$ is continuous in $[0, \infty)$.

To simplify the evaluation of integrals involving multiple lives, there are several well-known assumptions about the survival distribution of the group. In 1825, Gompertz introduced the following important type.

Assumption A. The force of mortality for the group follows Gompertz's law, i.e., $\mu(x) = Bc^x$ with $B > 0$ and $c > 1$.

In this case, we can substitute for the joint-life status (xy) by a single-life status (w) satisfying $\mu_w(t) = \mu_{xy}(t)$, where w is determined by the equality

$$c^x + c^y = c^w.$$

We refer to reference [2] for details. Therefore, we can simplify the joint-life status to the single-life status. However, the existence of such single-life status in the general case is not a known fact. So we state the following

Assumption B. In one survivorship group, for every two lives (x) and (y) , there exists a single-life status (w) in the group, such that $T(w)$'s distribution equals $T(xy)$'.

It is known that Assumption A implies Assumption B. In this paper we will prove a striking result, which claims that the converse is true.

Theorem 1.1. *Assumption B implies Assumption A.*

Therefore, Assumption A is equivalent to Assumption B. This means, the assumption that mortality for each life follows Gompertz's law is the necessary and sufficient criterion to guarantee that the joint-life status can be substituted for by a single-life status which has the same force of mortality as that of the joint-life status.

For the usage of Gompertz's law, we refer readers to references [3] ,[4] and [6]

2 Proof of Theorem 1.1

For the single-life statue (x) and (y), their force of mortality are $\mu(x+t)$ and $\mu(y+t)$. Since the force of mortality for the joint-life status is the sum of the forces of mortality for the associated lives, the force of mortality $\mu_{xy}(t)$ of the joint-life status (xy) is equal to

$\mu(x+t)+\mu(y+t)$. Then, Assumption (B) is equivalent to that there exists a single-life status (w) such that

$$\mu(x+t)+\mu(y+t)=\mu(w+t), \quad t\in[0,\infty). \tag{2.1}$$

Now we formulate Theorem 1.1 into the following mathematical proposition.

Theorem 2.1. *Suppose that $\mu:[0,+\infty)\to[0,+\infty)$ is a nonnegative and continuous function satisfying $\mu\not\equiv 0$. If there exists a function $w:[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ such that for every $x,y,t\geq 0$,*

$$\mu(x+t)+\mu(y+t)=\mu\big(w(x,y)+t\big). \tag{2.2}$$

Then for every $x,y\geq 0$,

$$\mu(x)=Bc^x, \quad w(x,y)=\frac{\ln(c^x+c^y)}{\ln c} \tag{2.3}$$

with constants $B>0$, $c>1$.

Remark 2.2. It is obvious that the function defined in (2.3) satisfies

$$\int_0^\infty \mu(x)dx=\infty,$$

which is the necessary and sufficient condition for μ to be the force of mortality.

Before proving Theorem 2.1, we need several technical lemmas.

Lemma 2.3. *Suppose that $g\in C[0,+\infty)$ satisfies, for every $x,y>0$,*

$$g(x+y)=g(x)\cdot g(y) \tag{2.4}$$

and $g(x)\not\equiv 0$. Then for every $x\geq 0$,

$$g(x)=c^x \quad \textit{with} \;\; c=g(1). \tag{2.5}$$

This is a known result of Advanced Calculus, and we omit the proof.

Lemma 2.4. *Suppose that $\mu(x)$ and $w(x,y)$ are the two functions defined in Theorem 2.1. Then we have*
(i) *$\mu(x)$ is unbounded on $[0,+\infty)$;*
(ii) *$w(0,0)>0$ and $w(x,y)$ is continuous on $[0,+\infty)\times[0,+\infty)$.*

Proof.
(i) Since $\mu\not\equiv 0$, we find a point $z_0\geq 0$ such that $\mu(z_0)>0$. Choosing $x=y=0$ in (2.2), we have that for every $t\geq 0$,

$$2\mu(t)=\mu\big(w(0,0)+t\big). \tag{2.6}$$

If $w(0,0)=0$, then $\mu(t)\equiv 0$. This contradicts our assumption $\mu\not\equiv 0$. Therefore, we have $w(0,0)>0$. And it follows from (2.6) that

$$2^m\mu(z_0)=\mu\big(mw(0,0)+z_0\big) \tag{2.7}$$

where m is any positive integer. Denote $z_m=mw(0,0)+z_0$. Letting $m\to+\infty$ in (2.6), we find that $\mu(z_m)\to+\infty$. Therefore, the function $\mu(x)$ is unbounded.
(ii) For every $x_0,y_0\geq 0$, we will prove that

$$\lim_{(x,y)\to(x_0,y_0)} w(x,y)=w(x_0,y_0).$$

Fix (x_0,y_0) and denote

$$d^+=\limsup_{(x,y)\to(x_0,y_0)} w(x,y),\quad d^-=\liminf_{(x,y)\to(x_0,y_0)} w(x,y),\quad d_0=w(x_0,y_0).$$

If we prove that $d^+=d_0=d^-$, we will finish the proof of (ii). First we show that $d^+<+\infty$. By the definition of d^+, we may find two sequences $\{x_n\}$, $\{y_n\}$, $n=1,\dots,$ such that

$$x_n\to x_0,\quad y_n\to y_0,\quad w(x_n,y_n)=d_n\to d^+.$$

Let $z_0\geq 0$ be as in the proof of (i). Noting that

$$\mu(x_n+t)+\mu(y_n+t)=\mu(d_n+t) \tag{2.8}$$

for every $t\geq 0$ and choosing

$$t=t_n=\Big([\frac{d_n}{w(0,0)}]+1\Big)w(0,0)-d_n+z_0$$

in (2.8), we have

$$\mu(x_n + t_n) + \mu(y_n + t_n) = \mu\Big(\big([\frac{d_n}{w(0,0)}] + 1\big)w(0,0) + z_0\Big). \tag{2.9}$$

Here $[d_n/w(0,0)]$ denotes the integral part of the number $d_n/w(0,0)$. By the definition of t_n, we find that $t_n \leq w(0,0) + z_0$. Therefore the sequences $\{x_n + t_n\}$ and $\{y_n + t_n\}$ are bounded. Recalling (2.7) and (2.9), we deduce that

$$\mu(x_n + t_n) + \mu(y_n + t_n) = 2^{[d_n/w(0,0)]+1}\mu(z_0). \tag{2.10}$$

If $d_n \to +\infty$, then the right hand side tends to $+\infty$ while the left hand side of (2.10) remains bounded. That is impossible. Therefore $d^+ < +\infty$.

Now we prove that $d^+ = d^- = d_0$.

Recalling (2.2) and letting $n \to +\infty$ in (2.8), we conclude that

$$\mu(d_0 + t) = \mu\big(w(x_0, y_0) + t\big) = \mu(x_0 + t) + \mu(y_0 + t) = \mu(d^+ + t) \tag{2.11}$$

for every $t \geq 0$. If $d^+ \neq d_0$, without loss of generality, we assume that $d^+ > d_0$. Then it follows from (2.11) that

$$\mu(d^+ - d_0 + s) = \mu(s)$$

for every $s \geq d_0$. Therefore $\mu(s)$ is a periodic function on $[d_0, +\infty)$. That implies that $\mu(x)$ is bounded in $[0, +\infty)$. This contradicts (i). Therefore, we have $d^+ = d_0$. Since $d^- \leq d^+ < \infty$, by the same reasoning as above we obtain that $d^- = d_0 = \mu(x_0, y_0)$. Therefore the function $w(x, y)$ is continuous at (x_0, y_0).

In the following we assume that

$$\mu \in C^1(0, +\infty). \tag{2.12}$$

And later we will prove that this assumption is redundant by an approximation argument.

Lemma 2.5. *Suppose that $\mu(x)$ and $w(x, y)$ are determined in Theorem 2.1 and moreover, $\mu(x)$ satisfies* (2.12). *Then $\partial w/\partial x$ and $\partial w/\partial y$ exist for every $x, y > 0$ and belong to $C\big([0, +\infty) \times [0, \infty)\big)$.*

Proof. Let $x_0, y_0 \geq 0$ be fixed. We assert that there exists a point $z_0 > w(x_0, y_0)$ such that $\mu'(z_0) \neq 0$. Otherwise, $\mu'(z) \equiv 0$ for $z > w(x_0, y_0)$. That means $\mu(z)$ is a constant when

$z > w(x_0, y_0)$. Since $\mu(z)$ is continuous on $[0, w(x_0, y_0)]$, it follows that $\mu(z)$ is bounded on $[0, +\infty)$. That is impossible because it contradicts (i) in Lemma 2.4.

Now we choose $t = z_0 - w(x_0, y_0)$ in (2.2) to obtain

$$\mu\big(x + z_0 - w(x_0, y_0)\big) + \mu\big(y + z_0 - w(x_0, y_0)\big) = \mu\big(w(x, y) + z_0 - w(x_0, y_0)\big).$$

Since

$$\mu'\big(w(x, y) + z_0 - w(x_0, y_0)\big)\big|_{x=x_0, y=y_0} = \mu'(z_0) \neq 0$$

and $\mu'(x)$, $w(x, y)$ are continuous, by the Implicit Function Theorem there exists a $\delta > 0$ such that

$$\begin{aligned} w(x, y) =& \mu^{-1}\Big(\mu\big(x + z_0 - w(x_0, y_0)\big) + \mu\big(y + z_0 - w(x_0, y_0)\big)\Big) \\ &+ w(x_0, y_0) - z_0 \end{aligned} \tag{2.13}$$

for all x, y satisfying $|x - x_0| + |y - y_0| < \delta$. Because all functions involved in the right hand side of (2.13) are C^1-differentiable, $w(x, y)$ is C^1-differentiable at (x_0, y_0) too.

Lemma 2.6. *Suppose that $\mu(x)$ and $w(x, y)$ are determined in Theorem 2.1 and moreover, $\mu(x)$ satisfies* (2.12). *Then there exist two constants $B > 0$, $c > 1$ such that for every $x, y \geq 0$,*

$$\mu(x) = Bc^x, \quad w(x, y) = \frac{\ln(c^x + c^y)}{\ln c}.$$

Proof. Set $y = 0$ in (2.2) and denote $f(x) = w(x, 0)$. Then

$$\mu(x + t) + \mu(t) = \mu\big(w(x, 0) + t\big) = \mu\big(f(x) + t\big)$$

where $x, t \geq 0$. That is, for every $x, y > 0$,

$$\mu(x + y) + \mu(y) = \mu\big(f(x) + y\big)$$

where $f \in C^1[0, +\infty)$ by Lemma 2.5. Differentiating with respect to x and y, we obtain

$$\mu'(x + y) = \mu'\big(f(x) + y\big)f'(x), \quad \mu'(x + y) + \mu'(y) = \mu'\big(f(x) + y\big)$$

for every $x, y > 0$.

Canceling the same term $\mu'\big(f(x) + y\big)$ in the above equalities, we conclude that, for every $x, y > 0$,

$$\mu'(y)f'(x) = \mu'(x + y)\big(1 - f'(x)\big). \tag{2.14}$$

Exchanging the variables x, y in (2.14) we have

$$\mu'(x)f'(y) = \mu'(x+y)\big(1 - f'(y)\big). \tag{2.15}$$

We first assert that for every $x > 0$,

$$f'(x) \neq 1, \quad f'(x) \neq 0. \tag{2.16}$$

Otherwise, there exists a point $x_0 > 0$ such that $f'(x_0) = 1$ or $f'(x_0) = 0$. In the first case, it follows from (2.14) with $x = x_0$ that $\mu'(y) = 0$ for every $y > 0$. This implies that $\mu(x)$ is a constant. This contradicts (i) in Lemma 2.4. In the second case, it follows from (2.14) with $x = x_0$ that $\mu'(x_0 + y) = 0$ for every $y > 0$. This fact and the continuity $\mu(x)$ imply that $\mu(x)$ is bounded, too. This also leads to a contradiction to (i) in Lemma 2.4. Therefore, (2.16) holds. Using the same argument we can easily show that for every $x > 0$,

$$\mu'(x) \neq 0. \tag{2.17}$$

Now denote

$$g(x) = \frac{f'(x)}{1 - f'(x)}.$$

It follows from (2.14) and (2.15) that

$$\mu'(x+y) = g(y)\mu'(x) = g(x)\mu'(y), \tag{2.18}$$

which implies that for every $x, y > 0$,

$$\frac{\mu'(x)}{g(x)} = \frac{\mu'(y)}{g(y)} = C \tag{2.19}$$

with the constant $C \neq 0$. By virtue of (2.18), we obtain

$$g(x+y) = g(x)g(y).$$

Since $g(x) \neq 0$, we apply Lemma 2.3 to conclude that $g(x) = c^x$ with $c = g(1) > 0$. Solving the ordinary differential equation (2.19), we deduce, for every $x > 0$,

$$\mu(x) = A + Bc^x,$$

where A, B are two constants. Choosing $x = y = 0$ in (2.2) we have that for every $t \geq 0$,

$$2(A + Bc^t) = 2\mu(t) = \mu\big(w(0,0) + t\big) = A + Bc^{w(0,0)+t},$$

which implies $A = 0$ and $c = 2^{\frac{1}{w(0,0)}}$. Since $\mu(x) \geq 0$ is unbounded, $B > 0$ and $c > 1$ follows. By the direct calculation we obtain that $w(x, y) = \ln(c^x + c^y)/\ln c$ and complete the proof of the lemma.

Now we are in the position to prove Theorem 2.1. Our strategy is to regularize the function $\mu(x)$ in Theorem 2.1, then use Lemma 2.6 to determine the approximation functions and finally take the limit to obtain the result.

Proof of Theorem 2.1. Now let $\phi(x) \in C_0^\infty$ be a mollifier in $(-\infty, +\infty)$. That means, $\phi \geq 0$, $\operatorname{supp}\phi \subset (-1, 1)$ and $\int_{-1}^{1} \phi(x)\,dx = 1$. For every ε, $0 < \varepsilon \leq 1$, we set

$$\phi_\varepsilon(x) = \frac{1}{\varepsilon}\phi\big(\frac{x}{\varepsilon}\big),$$

and define

$$\mu_\varepsilon(x) = \int_{-\varepsilon}^{\varepsilon} u(x - z)\phi_\varepsilon(z)\,dz = \int_{x-\varepsilon}^{x+\varepsilon} \mu(z)\phi_\varepsilon(x - z)\,dz$$

for every $x \geq \varepsilon$. It is obvious that $\mu_\varepsilon \in C^\infty[\varepsilon, +\infty)$. It follows from (2.2) that, for every $t \geq \varepsilon$ and $x, y \geq 0$,

$$\mu_\varepsilon(x + t) + \mu_\varepsilon(y + t) = \mu_\varepsilon\big(w(x, y) + t\big). \tag{2.20}$$

For every $x \geq 0$, define

$$\nu_\varepsilon(x) = \mu_\varepsilon(x + \varepsilon).$$

It follows from (2.20) that, for every $x, y, t \geq 0$

$$\nu_\varepsilon(x + t) + \nu_\varepsilon(y + t) = \nu_\varepsilon\big(w(x, y) + t\big).$$

Moreover, ν_ε satisfies (2.12). By virtue of Lemma 2.5 we conclude that there exist $\bar{B}_\varepsilon > 0$ and $c_\varepsilon > 1$ such that

$$\nu_\varepsilon(x) = \bar{B}_\varepsilon c_\varepsilon^x.$$

Thus we have that for every $x \geq \varepsilon$,

$$\mu_\varepsilon(x) = \nu_\varepsilon(x - \varepsilon) = \bar{B}_\varepsilon c_\varepsilon^{x-\varepsilon} = B_\varepsilon c_\varepsilon^x$$

with $B_\varepsilon = \bar{B}_\varepsilon c_\varepsilon^{-\varepsilon}$. Fix any point $x \in (0, +\infty)$. For every $\varepsilon \le x$, $\mu_\varepsilon(x)$ makes sense. Recalling the continuity of $\mu(x)$, we have that for every $x > 0$,

$$\mu_\varepsilon(x) \to \mu(x), \quad \text{as } \varepsilon \to 0.$$

That is, for $x \in (0, +\infty)$,

$$B_\varepsilon c_\varepsilon^x \to \mu(x), \quad \text{as } \varepsilon \to 0.$$

Let $z_0 \ge 0$ be a point such that $\mu(z_0) > 0$. It follows from (2.6) and (2.7) that

$$2\mu(z_0) = \mu\big(w(0,0) + z_0\big), \quad 4\mu(z_0) = \mu\big(2w(0,0) + z_0\big). \tag{2.21}$$

It follows from (2.21) that

$$c_\varepsilon^{w(0,0)} = \frac{\mu_\varepsilon\big(2w(0,0) + z_0\big)}{\mu_\varepsilon\big(w(0,0) + z_0\big)} \to \frac{\mu(2w(0,0) + z_0)}{\mu\big(w(0,0) + z_0\big)} = 2,$$

which implies that

$$c_\varepsilon \to c = 2^{\frac{1}{w(0,0)}}.$$

And we have

$$B_\varepsilon = \frac{\mu_\varepsilon(z_0)}{c_\varepsilon^{z_0}} \to \frac{\mu(z_0)}{c^{z_0}} = B > 0.$$

Therefore $\mu(x) = Bc^x$ for every $x > 0$. Because $\mu(x)$ is continuous at x=0, the first equality in (2.3) holds for every $x \ge 0$. And the second equality follows by the direct calculation. As $\mu(x) \ge 0$ is unbounded, $c > 1$ follows. Therefore we complete the proof of Theorem 2.1.

Based on Theorem 2.1, Theorem 1.1 follows.

References

1. Black K Jr and Skipper H D Jr. Life Insurance. Prentice-Hall, Inc.,1994.
2. Bowers N L, Gerber H U, Hickman J C, Jones D A, and Nesbitt C J. Actuarial Mathematics. Schaumburg, Illinois: The Society of Actuaries,1997.
3. Brown R L. Introduction to Mathematics of Demography. ACTEX Publications,1997.
4. London D. Graduation: The Revision of Estimates. ACTEX Publications,1995.
5. London D. Survival Models and Their Estimation. ACTEX Publications,1997.
6. Pollard J H. Heterogeneity, dependence among causes of death and Gompertzs *Mathematical Population Studies,* 1993, 4(2): 117–132.

7. Pollard J H and Valkovics E J. The Gompertz distribution and its appplications. *Genus,* 1992, XLVIII(3-4):15-28.
8. Pollard J H and Valkovics E J. On the use of the truncated Gompertz distribution and other models to represent the parity progression functions of high fertility populations. *Mathematical Population Studies,* 1997, 6(4): 291-305.