5. Fibration formula

Inspired by the induction formula for the characteristic classes [28, Proposition 5.3.7], we prove the following fibration formula.

Theorem 5.1 (Fibration formula). *Under the notation and conditions (C1)-(C3) in 4.1, we have*

$$C_{X/S}(\mathcal{F}) = \delta^! C_{X/Y}(\mathcal{F}) + C^Z_{X/Y/S}(\mathcal{F}) \quad \text{in} \quad H^0(X, \mathcal{K}_{X/S}), \tag{5.1.1}$$

where $\delta^! : H^0(X, \mathcal{K}_{X/Y}) \to H^0(X, \mathcal{K}_{X/S})$ *is defined in* (3.12.3).

When $g : Y \to S$ is smooth of relative dimension r, we have $\delta^! = c_r(f^*\Omega^{1,\vee}_{Y/S})$ by (3.13.2). Then (5.1.1) can be rewritten as

$$C_{X/S}(\mathcal{F}) = c_r(f^*\Omega^{1,\vee}_{Y/S}) \cap C_{X/Y}(\mathcal{F}) + C^Z_{X/Y/S}(\mathcal{F}) \quad \text{in} \quad H^0(X, \mathcal{K}_{X/S}). \tag{5.1.2}$$

In the case that $X = Y \to S = \mathrm{Spec} k$ is smooth over a field k. Since $\mathrm{id} : X\backslash Z \to X\backslash Z$ is universally locally acyclic relatively to $\mathcal{F}|_{X\backslash Z}$, the cohomology sheaves of $\mathcal{F}|_{X\backslash Z}$ are locally constant on $X\backslash Z$. In this case, the class $C^Z_{X/Y/S}(\mathcal{F})$ is Abbes-Saito's localized characteristic class [1, Definition 5.2.1] and (5.1.1) follows from [1, Proposition 5.2.3].

We will prove Theorem 5.1 in 5.10. This proof is based on Lemma 5.5 below, which is due to an anonymous referee.

5.2. Let us recall a lifting result in ∞-category. Let $\mathcal{C}$ be a stable ∞-category. A triangle in $\mathcal{C}$ is a functor $F : \Delta^1 \times \Delta^1 \to \mathcal{C}$ (a square in $\mathcal{C}$) such that $F(1,0)$ is a zero object in $\mathcal{C}$. Usually, we write a triangle F as $F(0,0) \to F(0,1) \to F(1,1)$ or as a coherent commutative diagram in $\mathcal{C}$

$$\begin{array}{ccc} F(0,0) & \longrightarrow & F(0,1) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & F(1,1). \end{array} \tag{5.2.1}$$

Let $\mathrm{Tri}(\mathcal{C}) \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ be the full sub-∞-category spanned by triangles in $\mathcal{C}$. Let $\mathrm{ExTri}(\mathcal{C}) \subseteq \mathrm{Tri}(\mathcal{C})$ be the full sub-∞-category spanned by exact triangles (cofiber sequences) in $\mathcal{C}$. Let $\theta : \mathrm{ExTri}(\mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ be the functor sending a cofiber sequence $P'' \to P \to P'$ to $P \to P'$, which is a trivial Kan fibration by [20, Remark 1.1.1.7].

Consider a commutative diagram in $\mathcal{C}$ between cofiber sequences

$$\begin{array}{ccccc} P'' & \longrightarrow & P & \longrightarrow & P' \\ & & \downarrow & & \downarrow \\ Q'' & \longrightarrow & Q & \longrightarrow & Q'. \end{array} \tag{5.2.2}$$

We view the right square as a 1-simplex $\Delta^1 \to \mathrm{Fun}(\Delta^1, \mathcal{C})$. Since θ is a trivial Kan fibration, there is a lifting $\Delta^1 \to \mathrm{ExTri}(\mathcal{C})$, which is unique up to a contractible space, making the following diagram

$$\begin{array}{ccc} \partial\Delta^1 & \longrightarrow & \mathrm{ExTri}(\mathcal{C}) \\ \downarrow & \nearrow & \downarrow \theta \\ \Delta^1 & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathcal{C}) \end{array} \tag{5.2.3}$$

commutes. Thus there is a morphism $P'' \to Q''$ such that the following diagram commutes:

$$\begin{array}{ccccc} P'' & \longrightarrow & P & \longrightarrow & P' \\ \downarrow & & \downarrow & & \downarrow \\ Q'' & \longrightarrow & Q & \longrightarrow & Q'. \end{array} \tag{5.2.4}$$

Now for a commutative diagram (with solid arrows) between three cofiber sequences

(5.2.5)
$$\begin{tikzcd} & P'' \arrow[r] \arrow[d, dashed] \arrow[dl, dashed] & P \arrow[r] \arrow[d] \arrow[dl] & P' \arrow[d] \arrow[dl] \\ R'' \arrow[r] & Q'' \arrow[r] \arrow[l, dashed] & Q \arrow[r] \arrow[l] & Q' \arrow[l] \\ & R \arrow[r] & R', & \end{tikzcd}$$

by (5.2.4), we may find dashed arrows $P'' \dashrightarrow Q'', Q'' \dashrightarrow R''$ and $P'' \dashrightarrow R''$ such that any of the three lateral faces of the triangular prism (5.2.5) are commutative. We claim that the triangle

(5.2.6)
$$\begin{tikzcd} & P'' \arrow[dl, dashed] \arrow[d, dashed] \\ R'' & Q'' \arrow[l, dashed] \end{tikzcd}$$

formed by dashed arrows in (5.2.5) is also commutative. Indeed the right-smaller triangular prism in (5.2.5) defines a 2-simplex $\Delta^2 \to \mathrm{Fun}(\Delta^1, \mathcal{C})$, again by the triviality of θ, there is a lifting $\Delta^2 \to \mathrm{ExTri}(\mathcal{C})$ making

(5.2.7)
$$\begin{tikzcd} \partial\Delta^2 \arrow[r] \arrow[d] & \mathrm{ExTri}(\mathcal{C}) \arrow[d, "\theta"] \\ \Delta^2 \arrow[r] \arrow[ur, dashed] & \mathrm{Fun}(\Delta^1, \mathcal{C}) \end{tikzcd}$$

into a commutative diagram. This proves the claim. Note that if $\mathcal{C}$ is only a triangulated category, even though the three dashed arrows may still exist such that any of the three lateral faces of the triangular prism (5.2.5) are commutative, but the diagram (5.2.6) may not commute.

5.3. Note that there is a canonical functor

(5.3.1)
$$\mathrm{FD} : \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}) \to \mathrm{Tri}(\mathcal{C})$$

sending a square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "p"] \\ Z \arrow[r, "q"'] & W \end{tikzcd}$$

in $\mathcal{C}$ to the following triangle

$$\begin{tikzcd} X \arrow[r, "{(f,g)}"] \arrow[d] & Y \oplus Z \arrow[d, "p-q"] \\ 0 \arrow[r] & W. \end{tikzcd}$$

In the following, a fiber sequence in the stable ∞-category $\mathrm{ExTri}(\mathcal{C})$ will be called a nine diagram in $\mathcal{C}$.

Lemma 5.4. *Let $\mathcal{C}$ be a stable ∞-category. There is a functor*

(5.4.1)
$$T : \mathrm{ExTri}(\mathrm{ExTri}(\mathcal{C})) \to \mathrm{Fun}(\Delta^1, \mathcal{C})$$

sending a nine diagram

(5.4.2)
$$\begin{array}{ccccc} K^{00} & \longrightarrow & K^{01} & \longrightarrow & K^{02} \\ \downarrow & & \downarrow & & \downarrow \\ K^{10} & \longrightarrow & K^{11} & \longrightarrow & K^{12} \\ \downarrow & & \downarrow & & \downarrow \\ K^{20} & \longrightarrow & K^{21} & \longrightarrow & K^{22} \end{array}$$

to $\mathrm{cofib}(K^{00} \to K^{01} \oplus K^{10}) \to K^{02} \oplus K^{11} \oplus K^{20}$.

Proof. Using the functor FD (cf. (5.3.1)) three times, we can get a functor

(5.4.3)
$$\mathrm{ExTri}(\mathrm{ExTri}(\mathcal{C})) \to \mathrm{Tri}(\mathrm{Tri}(\mathcal{C}))$$

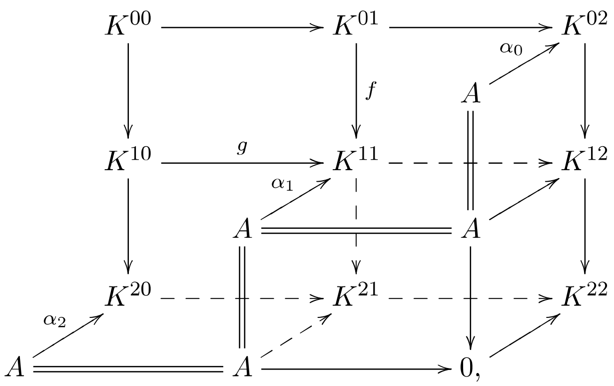
sending a nine diagram (5.4.2) to the following commutative diagram

(5.4.4)
$$\begin{array}{ccccc} K^{00} & \longrightarrow & K^{01} \bigoplus K^{10} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & K^{02} \bigoplus K^{11} \bigoplus K^{20} & \longrightarrow & K^{12} \bigoplus K^{21} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & K^{22}. \end{array}$$

This immediately induces the required functor (5.4.1). □

The following Lemma is due to an anonymous referee.

Lemma 5.5. *Let $\mathcal{C}$ be a stable ∞-category. Consider a commutative diagram in $\mathcal{C}$:*

(5.5.1)


where the back face is a nine diagram, the right face and the lower face are maps of fiber sequences. Assume that $\mathrm{Hom}_{h\mathcal{C}}(A, K^{00}) = \mathrm{Hom}_{h\mathcal{C}}(A, \Sigma K^{00}) = 0$*. Let* $\beta_0 \in \mathrm{Hom}_{h\mathcal{C}}(A, K^{01})$ *be the unique lifting of* α_0 *and* $\beta_2 \in \mathrm{Hom}_{h\mathcal{C}}(A, K^{10})$ *the unique lifting of* α_2*. Then we have*

(5.5.2)
$$\alpha_1 = f \circ \beta_0 + g \circ \beta_2 \quad \text{in} \quad \mathrm{Hom}_{h\mathcal{C}}(A, K^{11}).$$

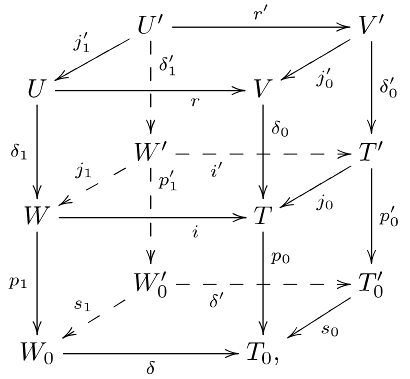
Proof. Applying 5.2 six times, we can complete the diagram (5.5.1) into a map of nine diagrams. By Lemma 5.4, we get a commutative diagram

(5.5.3)
$$\begin{array}{ccc} A & \longrightarrow & A\oplus A\oplus A \\ \beta\big\downarrow & (*) & \big\downarrow(\alpha_0,\alpha_1,\alpha_2) \\ Z & \longrightarrow & K^{02}\oplus K^{11}\oplus K^{20}, \end{array}$$

where $Z = \mathrm{cofib}(K^{00} \to K^{01} \oplus K^{10})$. Let $\beta' \in \mathrm{Hom}_{h\mathcal{C}}(A, K^{01} \oplus K^{10})$ be the unique lifting of β. Then the commutativity of the square $(*)$ in (5.5.3) implies the equalities $\beta' = (\beta_0, \beta_2)$ and $\alpha_1 = f \circ \beta_0 + g \circ \beta_2$ in $h\mathcal{C}$. □

From subsection 5.6 to 5.9, we will construct a similar diagram (5.5.1) for the non-acyclicity class in (5.10.3). Then Theorem 5.1 will follow directly from Lemma 5.5. Readers who are not interested in this construction can skip directly to subsection 5.10.

5.6. Let $\mathrm{Corr}_S^{\otimes} = \mathrm{Corr}_{\mathrm{sep;all}}^{\mathrm{proper}}(\mathrm{Sch}_S)$ be the symmetric monoidal $(\infty, 2)$-category of correspondences (cf. [10, Part III]). Let $\mathcal{D} : \mathrm{Corr}_S^{\otimes} \to \mathrm{Cat}_\infty$ be a 6-functor formalism of étale sheaves of Λ-modules on schemes over S such that $D(-,\Lambda) = h\mathcal{D}(-)$ is the homotopy category of $\mathcal{D}(-)$. Let $\mathcal{D}_{\mathrm{cons}}(-) \subseteq \mathcal{D}(-)$ be the full sub-∞-category spanned by perfect-constructible complexes. For any Noetherian scheme X over S, we have $D_{\mathrm{ctf}}(X,\Lambda) = h\mathcal{D}_{\mathrm{cons}}(X)$ (cf. [13, 7.2] and [12, §2]). Consider a commutative diagram in Sch_S

(5.6.1)


where, except for the squares on the left and right sides, all squares are cartesian. Let $\mathcal{F} \in \mathcal{D}_{\mathrm{cons}}(T)$ and $\mathcal{G} \in \mathcal{D}_{\mathrm{cons}}(T_0)$. Recall that we have a canonical morphism (cf. (2.1.3))

(5.6.2)
$$c_{\delta,p_0,\mathcal{F},\mathcal{G}} : i^*\mathcal{F} \otimes^L p_1^*\delta^!\mathcal{G} \to i^!(\mathcal{F} \otimes^L p_0^*\mathcal{G}).$$

By Lemma 2.3.(2), there is a commutative diagram for the face UW_0VT_0 in (5.6.1)

(5.6.3)
$$\begin{array}{ccc} \delta_1^*(i^*\mathcal{F}\otimes^L p_1^*\delta^!\mathcal{G}) & \xrightarrow{\delta_1^* c_{\delta,p_0,\mathcal{F},\mathcal{G}}} & \delta_1^* i^!(\mathcal{F}\otimes^L p_0^*\mathcal{G}) \\ \big\downarrow\simeq & & \text{b.c}\big\downarrow \delta_1^* i^! \to r^! \delta_0^* \\ r^*\delta_0^*\mathcal{F}\otimes^L \delta_1^* p_1^*\delta^!\mathcal{G} & \xrightarrow{c_{\delta,p_0\delta_0,\delta_0^*\mathcal{F},\mathcal{G}}} & r^!(\delta_0^*\mathcal{F}\otimes^L \delta_0^* p_0^*\mathcal{G}). \end{array}$$

Let $\mathcal{F}'=j_0^*\mathcal{F}$ and $\mathcal{G}'=s_0^*\mathcal{G}$. For the face $U'W_0'V'T_0'$ in (5.6.1), we have a commutative diagram

$$
\begin{array}{ccc}
\delta_1'^*(i'^*\mathcal{F}'\otimes^L p_1'^*\delta'^!\mathcal{G}') & \xrightarrow{\delta_1'^*c_{\delta',p_0',\mathcal{F}',\mathcal{G}'}} & \delta_1'^*i'^!(\mathcal{F}'\otimes^L p_0'^*\mathcal{G}') \\
\downarrow\simeq & & \text{b.c}\downarrow \delta_1'^*i'^!\to r'^!\delta_0'^* \\
r'^*\delta_0'^*\mathcal{F}'\otimes^L \delta_1'^*p_1'^*\delta'^!\mathcal{G}' & \xrightarrow{c_{\delta',p_0'\delta_0',\delta_0'^*\mathcal{F}',\mathcal{G}'}} & r'^!(\delta_0'^*\mathcal{F}'\otimes^L\delta_0'^*p_0'^*\mathcal{G}').
\end{array}
\tag{5.6.4}
$$

Lemma 5.7. *Consider the commutative diagram (5.6.1). Let $\mathcal{F}\in\mathcal{D}_{\mathrm{cons}}(T)$ and $\mathcal{G}\in\mathcal{D}_{\mathrm{cons}}(T_0)$. Put $\mathcal{F}'=j_0^*\mathcal{F}$ and $\mathcal{G}'=s_0^*\mathcal{G}$. There is a coherent commutative diagram $\Delta^1\times\Delta^1\times\Delta^1\to\mathcal{D}(U')$:*

$$
\begin{array}{ccccccc}
 & & j_1'^*\delta_1^*(i^*\mathcal{F}\otimes^L p_1^*\delta^!\mathcal{G}) & \longrightarrow & & & j_1'^*\delta_1^*i^!(\mathcal{F}\otimes^L p_0^*\mathcal{G}) \\
 & \swarrow^{\text{b.c}} & & & & \swarrow^{\text{b.c}} & \\
\delta_1'^*(i'^*\mathcal{F}'\otimes^L p_1'^*\delta'^!\mathcal{G}') & & \longrightarrow & & \delta_1'^*i'^!(\mathcal{F}'\otimes^L p_0'^*\mathcal{G}') & & \\
 & & \downarrow & & & & \downarrow \\
 & & j_1'^*r^*\delta_0^*\mathcal{F}\otimes^L j_1'^*\delta_1^*p_1^*\delta^!\mathcal{G} & \dashrightarrow & & & j_1'^*r^!(\delta_0^*\mathcal{F}\otimes^L\delta_0^*p_0^*\mathcal{G}) \\
\downarrow & \swarrow^{\text{b.c}} & & & \downarrow & \swarrow^{\text{b.c}} & \\
r'^*\delta_0'^*\mathcal{F}'\otimes^L\delta_1'^*p_1'^*\delta'^!\mathcal{G}' & & \longrightarrow & & r'^!(\delta_0'^*\mathcal{F}'\otimes^L\delta_0'^*p_0'^*\mathcal{G}'). & &
\end{array}
\tag{5.7.1}
$$

In the proof, we will use a result of the exponentiation of cocartesian fibrations [19, Proposition 3.1.2.1]. Let $\mathcal{E}\to\mathcal{C}$ be a cocartesian fibration of ∞-categories and K a simplicial set. Then $\mathrm{Fun}(K,\mathcal{E})\to\mathrm{Fun}(K,\mathcal{C})$ is a cocartesian fibration. An edge $f\xrightarrow{e}f'$ in $\mathrm{Fun}(K,\mathcal{E})$ is a cocartesian edge (for the fibration $\mathrm{Fun}(K,\mathcal{E})\to\mathrm{Fun}(K,\mathcal{C})$) if and only if for any vertex $v\in K$, $f(v)\xrightarrow{e(v)}f'(v)$ is a cocartesian edge in $\mathcal{E}$ (for the fibration $\mathcal{E}\to\mathcal{C}$).

The following proof is due to Jiangnan Xiong. The main idea is to give a universal characterization of $c_{\delta,p_0,\mathcal{F},\mathcal{G}}$ by using cocartesian/cartesian edges.

Proof. We recall some standard constructions. By unstraightening $\mathcal{D}$, we have a cocartesian fibration $\mathcal{C}_{S,\Lambda}^{\otimes}\to\mathrm{Corr}_S^{\otimes}$. In the following, we will denote a correspondence

$$
\begin{array}{ccc}
C & \to & X \\
\downarrow & & \\
Y & &
\end{array}
\tag{5.7.2}
$$

simply by $Y\leftarrow C\to X$. Objects in $\mathrm{Corr}_S^{\otimes}$ are given by $(X_1,\cdots,X_n)$ with $n\geqslant 0$ and $X_i\in\mathrm{Sch}_S$. Objects in $\mathcal{C}_{S,\Lambda}^{\otimes}$ are given by $(X_1,\cdots,X_n;\mathcal{F}_1,\cdots,\mathcal{F}_n)$ with $\mathcal{F}_i\in\mathcal{D}(X_i)$. We have an equivalence

$$
\mathcal{D}(X_1,\cdots,X_n)\xrightarrow{\simeq}\prod_{i=1}^n\mathcal{D}(X_i).
\tag{5.7.3}
$$

Let $\langle n\rangle=\{*,1,\cdots,n\}$. A morphism $(X_1,\cdots,X_n)\to(Y_1,\cdots,Y_m)$ in $\mathrm{Corr}_S^{\otimes}$ is given by a map $\alpha:\langle n\rangle\to\langle m\rangle$ with $\alpha(*)=*$ and correspondences

$$
Y_j\xleftarrow{\;g_j\;}C_j\xrightarrow{f_j=(f_{ji})}\prod_{i\in\alpha^{-1}(j)}X_i
\tag{5.7.4}
$$

for any $1\leqslant j\leqslant m$. For simplicity, we denote such a morphism by $(\alpha,g_!f^*)=(\alpha,(g_{j!}(f_{ji})^*))$ or by $g_!f^*=(g_{j!}(f_{ji})^*)$. The functor $\mathcal{D}(g_!f^*):\prod_{i=1}^n\mathcal{D}(X_i)\to\prod_{j=1}^m\mathcal{D}(Y_j)$ sends $(\mathcal{F}_1,\cdots,\mathcal{F}_n)$ to $(\mathcal{G}_1,\cdots,\mathcal{G}_m)$ with

$$
\mathcal{G}_j=g_{j!}f_j^*(\boxtimes^L_{i\in\alpha^{-1}(j)}\mathcal{F}_i)=g_{j!}(\otimes^L_{i\in\alpha^{-1}(j)}f_{ji}^*\mathcal{F}_i).
\tag{5.7.5}
$$

We have a cocartesian edge (for the fibration $\mathcal{C}^{\otimes}_{S,\Lambda} \to \mathrm{Corr}^{\otimes}_S$) above $g_!f^*$

$$(X_1,\cdots,X_n;\mathcal{F}_1,\cdots,\mathcal{F}_n) \to (Y_1,\cdots,Y_m;\mathcal{G}_1,\cdots,\mathcal{G}_m),$$

where $\mathcal{G}_i$ is defined by (5.7.5). If $m=n$ and $\alpha:\langle n\rangle \to \langle m\rangle$ is the identity, then the functor

$$\mathcal{D}(g_!f^*):\prod_{i=1}^{n}\mathcal{D}(X_i)\to\prod_{i=1}^{n}\mathcal{D}(Y_i);\quad (\mathcal{F}_i)\mapsto(g_{i!}f_i^*\mathcal{F}_i) \tag{5.7.6}$$

admits a right adjoint

$$\prod_{i=1}^{n}\mathcal{D}(Y_i)\to\prod_{i=1}^{n}\mathcal{D}(X_i);\quad (\mathcal{G}_i)\mapsto(f_{i*}g_i^!\mathcal{G}_i). \tag{5.7.7}$$

In this case, there is a locally cartesian edge

$$(X_1,\cdots,X_n;f_{1*}g_1^!\mathcal{G}_1,\cdots,f_{n*}g_n^!\mathcal{G}_n)\to(Y_1,\cdots,Y_n;\mathcal{G}_1,\cdots,\mathcal{G}_n)$$

above $g_!f^*:(X_1,\cdots,X_n)\to(Y_1,\cdots,Y_n)$ (cf. the dual version of [19, Corollary 5.2.2.5]).

Now we go back to the proof of Lemma 5.7. For convenience, we will use the arrow $\hookrightarrow$ (resp. $\twoheadrightarrow$) to indicate a locally cartesian edge (resp. cocartesian edge). We first give a universal characterization of $c_{\delta,p_0,\mathcal{F},\mathcal{G}}$. Indeed, we have a commutative diagram in $\mathrm{Corr}^{\otimes}_S$ (cf. the square WW_0TT_0 in (5.6.1)):

$$\begin{array}{ccccc} (W) & & \overline{\overline{\qquad\qquad\qquad}} & & (W) \\ {\scriptstyle (i,p_1)^*}\uparrow & & & & \downarrow{\scriptstyle i_!} \\ (T,W_0) & \xrightarrow{(\mathrm{id},\delta_!)} & (T,T_0) & \xrightarrow{(\mathrm{id},p_0)^*} & (T), \end{array} \tag{5.7.8}$$

where $(T,W_0)\xrightarrow{(i,p_1)^*}(W)$ is given by the correspondence $W=W\xrightarrow{(i,p_1)}T\times_S W_0$, $(W)\xrightarrow{i_!}(T)$ is given by the correspondence $T\xleftarrow{i}W=W$, $(T,W_0)\xrightarrow{(\mathrm{id},\delta_!)}(T,T_0)$ is given by $(T=T=T,T_0\xleftarrow{\delta}W_0=W_0)$ and $(T,T_0)\xrightarrow{(\mathrm{id},p_0)^*}(T)$ is given by $T=T\xrightarrow{(\mathrm{id},p_0)}T\times_S T_0$. The two compositions

$$(T,W_0)\to(W)=(W)\to(T)\quad\text{and}\quad(T,W_0)\to(T,T_0)\to(T)$$

are both equal to $i_!(i,p_1)^*$, i.e., the correspondence $T\xleftarrow{i}W\xrightarrow{(i,p_1)}T\times_S W_0$. Then we can characterize $c_{\delta,p_0,\mathcal{F},\mathcal{G}}$ by using the following lifting problem over (5.7.8):

$$\begin{array}{ccccc} (W;i^*\mathcal{F}\otimes^L p_1^*\delta^!\mathcal{G}) & & \overset{c_{\delta,p_0,\mathcal{F},\mathcal{G}}}{\dashrightarrow} & & (W;i^!(\mathcal{F}\otimes^L p_0^*\mathcal{G})) \\ {\scriptstyle\text{cocartesian}}\twoheaduparrow & & \overset{c'}{\searrow} & & \downarrow{\scriptstyle\text{cartesian}} \\ (T,W_0;\mathcal{F},\delta^!\mathcal{G}) & \hookrightarrow & (T,T_0;\mathcal{F},\mathcal{G}) & \twoheadrightarrow & (T,\mathcal{F}\otimes^L p_0^*\mathcal{G}). \end{array} \tag{5.7.9}$$

Indeed, the lifting c' uniquely exists by the property of cocartesian edges (cf. [19, 2.4.1.1]) and $c_{\delta,p_0,\mathcal{F},\mathcal{G}}$ is the unique morphism (up to a contractible space) making (5.7.9) commutes by the property of cartesian edges.

We denote the diagram (5.7.9) by $\Gamma_{WW_0TT_0}$. We also have three similar diagrams $\Gamma_{UW_0VT_0}$, $\Gamma_{W'W_0'T'T_0'}$ and $\Gamma_{U'W_0'V'T_0'}$. Now we prove Lemma 5.7 by constructing these four diagrams simultaneously. Here is a brief summary of the notation:

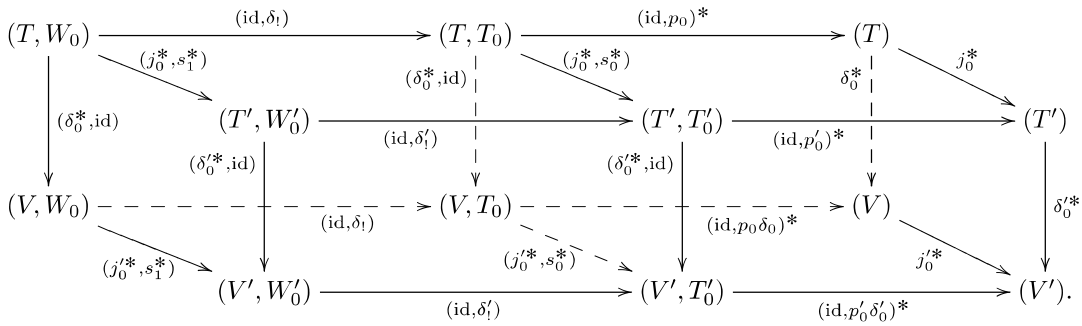
(1) The diagram $M:\Delta^2\to\mathrm{Fun}(\Delta^1\times\Delta^1,\mathcal{C}^{\otimes}_{S,\Lambda})$ (cf. (5.7.12)) realizes the bottom lines of these four diagrams at the same time.
(2) The left (resp. right) vertical arrows of these four diagrams are realized by the diagram $L:\Delta^1\to\mathrm{Fun}(\Delta^1\times\Delta^1,\mathcal{C}^{\otimes}_{S,\Lambda})$ (resp. $R:\Delta^1\to\mathrm{Fun}(\Delta^1\times\Delta^1,\mathcal{C}^{\otimes}_{S,\Lambda})$).
(3) The diagram $K:\Delta^1\to\mathrm{Fun}(\Delta^1\times\Delta^1,\mathcal{C}^{\otimes}_{S,\Lambda})$ realizes the diagonal dashed arrows (cf. c' in (5.7.9)) of these four diagrams.

(4) The top arrows (cf. $c_{\delta,p_0,\mathcal{F},\mathcal{G}}$ in (5.7.9)) of these four diagrams are realized by the diagram $F : \Delta^1 \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}^{\otimes}_{S,\Lambda})$ (cf. (5.7.14)). The diagram F determines the required commutative diagram (5.7.1) by pulling back to U'.

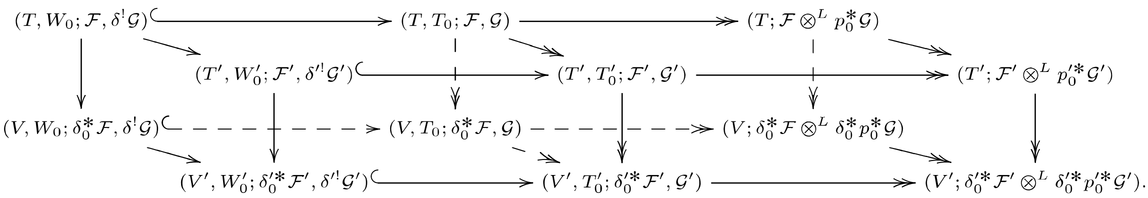
More precisely, let $M_1 : \Delta^0 \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}^{\otimes}_{S,\Lambda})$ be the commutative diagram

(5.7.10)
$$\begin{array}{ccc} (T, T_0; \mathcal{F}, \mathcal{G}) & \twoheadrightarrow & (T', T'_0; \mathcal{F}', \mathcal{G}') \\ \downarrow & & \downarrow \\ (V, T_0; \delta_0^* \mathcal{F}, \mathcal{G}) & \twoheadrightarrow & (V', T'_0; \delta_0'^* \mathcal{F}', \mathcal{G}'), \end{array}$$

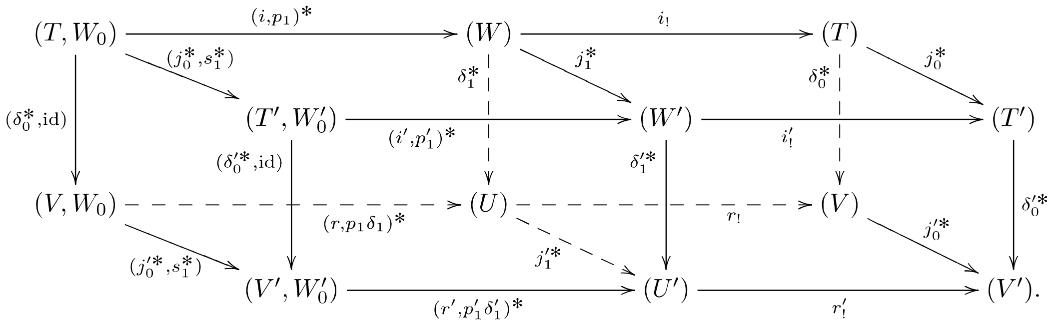
which is above $N_1 : \Delta^0 \cong \Delta^{\{1\}} \subset \Delta^2 \xrightarrow{N} \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathrm{Corr}^{\otimes}_S)$, where N is the following diagram:

(5.7.11)


Let $N_{01} = N|_{\Delta^{\{0,1\}}}$ be the left cubic diagram in (5.7.11) and $N_{12} = N|_{\Delta^{\{1,2\}}}$ the right cubic diagram. We can extend M_1 (cf. (5.7.10)) to a locally cartesian edge $M_{01} : \Delta^{\{0,1\}} \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}^{\otimes}_{S,\Lambda})$ above N_{01}, and extend M_1 to a cocartesian edge $M_{12} : \Delta^{\{1,2\}} \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}^{\otimes}_{S,\Lambda})$ above N_{12}. Now M_{01} and M_{12} give a diagram $\Lambda^2_1 \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}^{\otimes}_{S,\Lambda})$, which can be extended to a diagram $M : \Delta^2 \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}^{\otimes}_{S,\Lambda})$ as follows

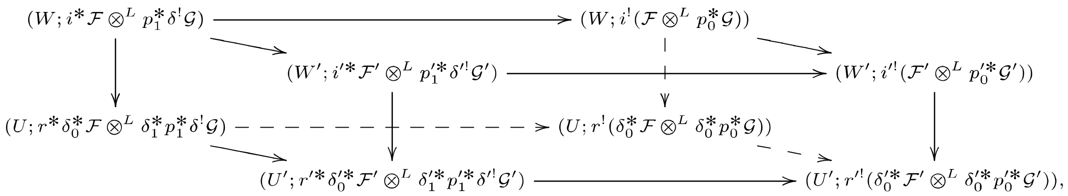
(5.7.12)


Now let $H : \Delta^2 \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathrm{Corr}^{\otimes}_S)$ be the following diagram

(5.7.13)


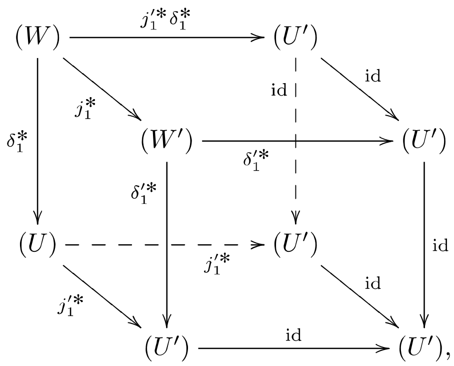
Note that $N_{02} = H_{02}$ (cf. (5.7.8)). In the following, we use the standard notation that $\Delta^2 \xrightarrow{(\alpha,\beta,\gamma)} \mathcal{C}$ means a functor $f : \Delta^2 \to \mathcal{C}$ such that $f|_{\Delta^{\{1,2\}}} = \alpha$, $f|_{\Delta^{\{0,2\}}} = \beta$ and $f|_{\Delta^{\{0,1\}}} = \gamma$. Let $L : \Delta^1 \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}^{\otimes}_{S,\Lambda})$ be a cocartesian lifting over H_{01} with $L_0 = M_0$. By the property of cocartesian edge, $\Lambda^2_0 \xrightarrow{(\bullet,M_{02},L)} \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}^{\otimes}_{S,\Lambda})$ can be extended to $\Delta^2 \xrightarrow{(K,M_{02},L)} \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}^{\otimes}_{S,\Lambda})$ over H. Let $R : \Delta^1 \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}^{\otimes}_{S,\Lambda})$ be a locally cartesian lifting over H_{12} with $R_1 = M_2$. Then $\Lambda^2_2 \xrightarrow{(R,K,\bullet)} \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}^{\otimes}_{S,\Lambda})$ can be extended to $\Delta^2 \xrightarrow{(R,K,F)} \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}^{\otimes}_{S,\Lambda})$. By construction, the diagram F is of the following form

(5.7.14)



which determines the required commutative diagram (5.7.1) by pulling back to U'. Indeed, let $A : \Delta^1 \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathrm{Corr}^{\otimes}_S)$ be the following diagram

(5.7.15)



where $A_0 = H_1$ and A_1 is the constant functor taking value (U'). Put $\mathrm{Fun}_{\mathcal{C}} = \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}^{\otimes}_{S,\Lambda})$ and $\mathrm{Fun}_S = \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathrm{Corr}^{\otimes}_S)$. Consider the covariant transport functor along A

(5.7.16) $$A_! : \mathrm{Fun}_{\mathcal{C}} \times_{\mathrm{Fun}_S} \{A_0\} \to \mathrm{Fun}_{\mathcal{C}} \times_{\mathrm{Fun}_S} \{A_1\} = \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{D}(U')).$$

The diagram F defines an edge $\Delta^1 \to \mathrm{Fun}_{\mathcal{C}} \times_{\mathrm{Fun}_S} \{A_0\}$. Now let $G = A_! \circ F : \Delta^1 \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{D}(U'))$, which defines a diagram $\Delta^1 \times \Delta^1 \times \Delta^1 \to \mathcal{D}(U')$. Unwinding the definitions, G is the required commutative diagram (5.7.1). □

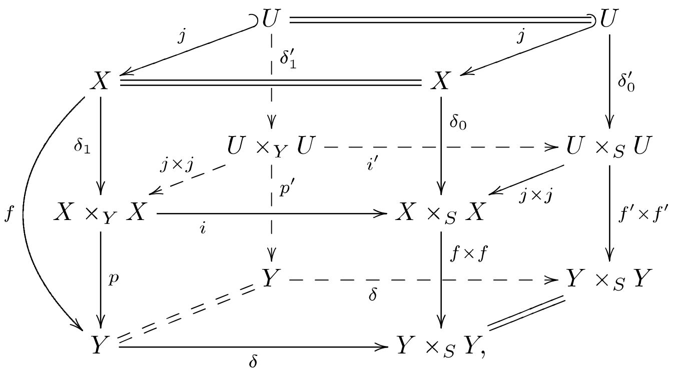
5.8. We apply Lemma 5.7 to construct the diagram (5.8.6) below, which is crucial for proving the fibration formula (5.1.1). Consider the diagrams (4.1.1) and (4.2.1). Let $j : U = X \backslash Z \to X$ be the open immersion and $\mathcal{F} \in D_{\mathrm{cons}}(X)$. We put

(5.8.1) $$\mathcal{H}_S = R\mathcal{H}om_{X \times_S X}(\mathrm{pr}_2^*\mathcal{F}, \mathrm{pr}_1^!\mathcal{F}), \qquad \mathcal{T}_S = \mathcal{F} \boxtimes^L_S D_{X/S}(\mathcal{F}),$$

(5.8.2) $$\mathcal{H}'_S = R\mathcal{H}om_{U \times_S U}(\mathrm{pr}_2^* j^*\mathcal{F}, \mathrm{pr}_1^! j^*\mathcal{F}), \qquad \mathcal{T}'_S = j^*\mathcal{F} \boxtimes^L_S D_{U/S}(j^*\mathcal{F}).$$

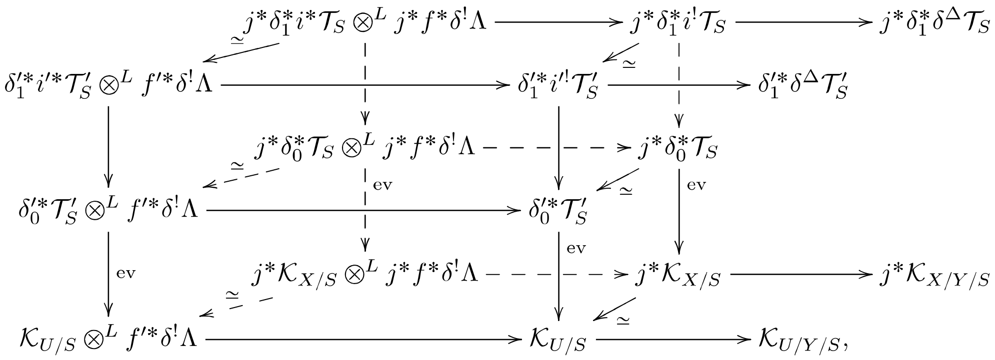
For later convenience, we form the following commutative diagram

(5.8.3)



where $f' = f|_U$, i' is the base change of δ, δ'_0 and δ'_1 are the diagonal morphisms. Then we have a commutative diagram

(5.8.4)

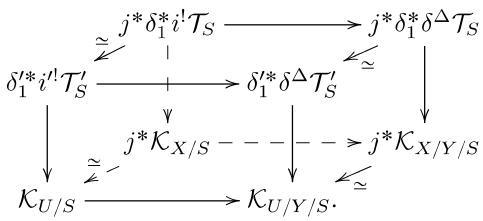


where the upper cube is obtained by applying Lemma 5.7 to the diagram (5.8.3), the lower cube is obtained by applying the following diagram to $j^* f^* \delta^! \Lambda \simeq f'^* \delta^! \Lambda \xrightarrow{\text{b.c}} \Lambda$:

$$
\begin{array}{ccc}
\delta_0'^* \mathcal{T}'_S \otimes^L - & \longleftarrow & j^* \delta_0^* \mathcal{T}_S \otimes^L - \\
\downarrow \text{ev} & & \downarrow \text{ev} \\
\mathcal{K}_{U/S} \otimes^L - & \longleftarrow & j^* \mathcal{K}_{X/S} \otimes^L - .
\end{array} \tag{5.8.5}
$$

Note that the horizontal rows in the top and bottom faces of (5.8.4) are cofiber sequences (cf. (2.11.1)). The diagram (5.8.4) further induces the following commutative diagram (by the same reason as in 5.2 or applying the dual version of 5.2 twice):

(5.8.6)



5.9. Using the 6-functor formalism $\mathcal{D} : \mathrm{Corr}_S^{\otimes} \to \mathrm{Cat}_\infty$, we can extend the definitions of $C_{X/S}$ (cf. (3.7.4) and (5.9.1)) and $C^Z_{X/Y/S}$ (cf. 4.5 and (5.9.5)) to objects in $\mathcal{D}_{\mathrm{cons}}(X)$ satisfying universal local acyclicity conditions[4]. We can also enhance the functor δ^Δ (cf. 2.11) and upgrade (4.2.5) to be a cofiber sequence in $\mathcal{D}(X)$. Some diagrams in Section 2 and Section 3 can be upgraded to coherent commutative diagrams in stable ∞-categories, for example, (3.13.9) and (4.10.1).

In the following, we assume the conditions (C1) and (C2) in 4.1 hold. We fix an object $\mathcal{F} \in \mathcal{D}_{\mathrm{cons}}(X)$ such that $X\backslash Z \to Y$ is universally locally acyclic relatively to $\mathcal{F}|_{X\backslash Z}$ and that $X \to S$ is universally locally acyclic relatively to $\mathcal{F}$. By the assumptions on $\mathcal{F}$, we have morphisms $C_{X/S}(\mathcal{F}) : \Lambda \to \mathcal{K}_{X/S}$ and $C_{U/Y}(\mathcal{F}) : \Lambda \to \mathcal{K}_{U/Y}$. More precisely, the morphism $C_{X/S}(\mathcal{F})$ is defined by the following composition[5]

(5.9.1) $$\Lambda \xrightarrow{\mathrm{id}} R\mathcal{H}om(\mathcal{F},\mathcal{F}) \xrightarrow[\simeq]{(3.1.5)} \delta_0^!\mathcal{H}_S \xleftarrow[\simeq]{(3.1.3)} \delta_0^!\mathcal{T}_S \to \delta_0^*\mathcal{T}_S \xrightarrow{\mathrm{ev}} \mathcal{K}_{X/S}.$$

Let $C_{X/Y/S}(\mathcal{F})$ be the following composition (cf. (4.10.1))

(5.9.2) $$\Lambda \to \delta_0^!\mathcal{H}_S \xleftarrow{\simeq} \delta_0^!\mathcal{T}_S \simeq \delta_1^! i^!\mathcal{T}_S \to \delta_1^* i^!\mathcal{T}_S \to \delta_1^*\delta^\Delta\mathcal{T}_S \xrightarrow{\mathrm{b.c}} \delta^\Delta\delta_0^*\mathcal{T}_S \xrightarrow{\mathrm{ev}} \delta^\Delta\mathcal{K}_{X/S} = \mathcal{K}_{X/Y/S},$$

which is equivalent to the composition $\Lambda \xrightarrow{C_{X/S}(\mathcal{F})} \mathcal{K}_{X/S} \xrightarrow{(4.2.5)} \mathcal{K}_{X/Y/S}$. Similar for $C_{U/Y/S}(\mathcal{F}) := C_{U/Y/S}(\mathcal{F}|_U)$. We have a commutative diagram

(5.9.3) $$\begin{array}{ccccc} \Lambda & \longrightarrow & j_*\Lambda & \longrightarrow & j_*\delta_1'^*\delta^\Delta\mathcal{T}_S' \simeq 0 \\ {\scriptstyle C_{X/Y/S}(\mathcal{F})}\downarrow & & {\scriptstyle C_{U/Y/S}(\mathcal{F}|_U)}\downarrow & \swarrow & \\ \mathcal{K}_{X/Y/S} & \longrightarrow & j_*\mathcal{K}_{U/Y/S}, & & \end{array}$$

where $\delta_1'^*\delta^\Delta\mathcal{T}_S' \simeq 0$ by (2.7.4) since $U \to Y$ is universally locally acyclic relatively to $\mathcal{F}|_U$. Applying $\tau_*\tau^! \to \mathrm{id} \to j_*j^*$ to (5.9.2), we get a commutative diagram between cofiber sequences

(5.9.4) $$\begin{array}{ccccc} \tau_*\tau^!\Lambda & \longrightarrow & \Lambda & \longrightarrow & j_*\Lambda \\ \downarrow & & \downarrow & & \downarrow \\ \tau_*\tau^!\delta_1^*i^!\mathcal{T}_S & \longrightarrow & \delta_1^*i^!\mathcal{T}_S & \longrightarrow & j_*\delta_1'^*i'^!\mathcal{T}_S' \\ \downarrow & & \downarrow & & \downarrow \\ \tau_*\tau^!\delta_1^*\delta^\Delta\mathcal{T}_S & \xrightarrow{\simeq} & \delta_1^*\delta^\Delta\mathcal{T}_S & \longrightarrow & j_*\delta_1'^*\delta^\Delta\mathcal{T}_S' \simeq 0 \\ \downarrow & & \downarrow & & \downarrow \\ \tau_*\tau^!\mathcal{K}_{X/Y/S} & \longrightarrow & \mathcal{K}_{X/Y/S} & \longrightarrow & j_*\mathcal{K}_{U/Y/S}. \end{array}$$

Note that the non-acyclicity class $C^Z_{X/Y/S}(\mathcal{F})$ is the composition (cf. 4.5)

(5.9.5) $$\Lambda \to \delta_1^*i^!\mathcal{T}_S \to \delta_1^*\delta^\Delta\mathcal{T}_S \xleftarrow{\simeq} \tau_*\tau^!\delta_1^*\delta^\Delta\mathcal{T}_S \to \tau_*\tau^!\mathcal{K}_{X/Y/S}$$

and (5.9.4) gives the following commutative diagram between cofiber sequences

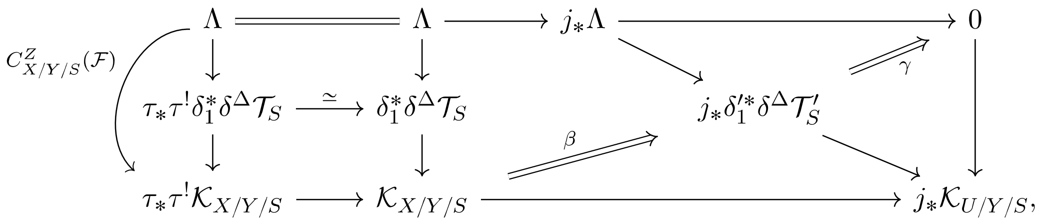
(5.9.6) $$\begin{array}{ccccc} \Lambda & = & \Lambda & \longrightarrow & 0 \\ {\scriptstyle C^Z_{X/Y/S}(\mathcal{F})}\downarrow & & \downarrow{\scriptstyle C_{X/Y/S}(\mathcal{F})} & & \downarrow \\ \tau_*\tau^!\mathcal{K}_{X/Y/S} & \longrightarrow & \mathcal{K}_{X/Y/S} & \longrightarrow & j_*\mathcal{K}_{U/Y/S}. \end{array}$$

[4]Let $\mathcal{C}^{\otimes}_{S,\Lambda} \to \mathrm{Corr}^{\otimes}_S$ be the unstraightening of $\mathcal{D} : \mathrm{Corr}^{\otimes}_S \to \mathrm{Cat}_\infty$. A morphism $X \to S$ is universally locally acyclic relatively to an object $\mathcal{F} \in \mathcal{D}_{\mathrm{cons}}(X)$ if and only if $(X,\mathcal{F})$ is dualisable in the symmetric monoidal ∞-category $\mathcal{C}^{\otimes}_{S,\Lambda}$ (cf. [18, Theorem 2.16] and [12, Definition 3.2]).

[5]In an ∞-category, the composition of two maps is well-defined up to a contractible space.

More precisely, it is given as follows

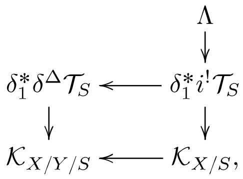
(5.9.7)



where β is the homotopy defined by the right squares in (5.9.4), γ is the homotopy determined by the isomorphism $j_*\delta_1'^*\delta^\Delta\mathcal{T}'_S \simeq 0$.

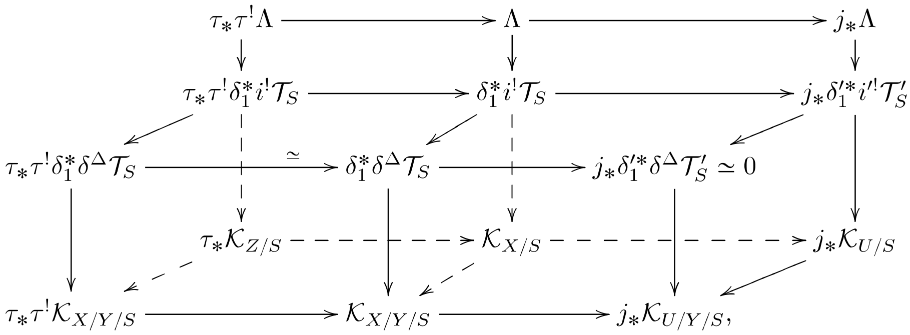
5.10. **Proof of Theorem 5.1.** Let $\mathcal{F} \in \mathcal{D}_{\mathrm{cons}}(X)$ such that the conditions (C1)-(C3) in 4.1 hold. Consider the notation in 4.1. Consider the diagram (5.8.3). Applying $\tau_*\tau^! \to \mathrm{id} \to j_*j^*$ to the following commutative diagram (cf. (4.10.1))

(5.10.1)



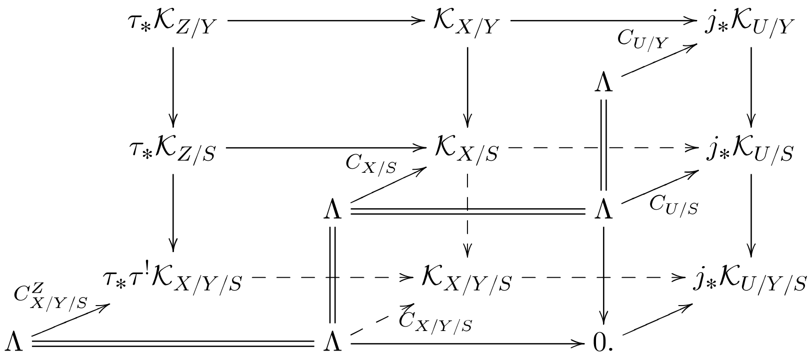
we get a commutative diagram between cofiber sequences

(5.10.2)



where the right diagram is obtained from the fact that j_*j^*(5.10.1) is isomorphic to the rightmost face by (5.8.6). By the construction of the non-acyclicity class $C^Z_{X/Y/S}$ (5.9.6) and (5.10.2), we get a coherent commutative diagram

(5.10.3)



Applying Lemma 5.5 to the above diagram, we get the fibration formula (5.1.1).