CHARACTERISTIC CYCLES AND NON-ACYCLICITY CLASSES FOR CONSTRUCTIBLE ETALE SHEAVES

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To the memory of Professor Linsheng Yin (1963–2015)

ABSTRACT. Let $f : X \to S$ be a morphism between smooth schemes over a perfect field k. Let $\mathcal{F}$ be a constructible sheaf on X and Z a closed subscheme of X such that f is $SS(\mathcal{F})$-transversal outside Z. We construct a class supported on the non-transversality locus Z by using the characteristic cycle $CC(\mathcal{F})$ defined by T.Saito. This class is a geometric counterpart of the non-acyclicity class introduced by the second author and Zhao in [15]. Under certain conditions, the formation of this class is compatible with base change and proper push-forward. It also satisfies the Milnor formula proved by Saito and a conductor formula. We conjecture that the image of this class under the cycle class map is the non-acyclicity class. This conjecture can be viewed as a (relative version of) Milnor type formula for non-isolated singularities.

We expect that this class can be lifted to a cycle supported on the cotangent bundle of X. For this, we introduce the singular support of $\mathcal{F}$ relatively to a morphism $f : X \to S$. When $f = id$, this goes back to Beilinson's definition of singular supports.

CONTENTS

1. INTRODUCTION

1.1. Let k be a perfect field of characteristic $p > 0$ and $S = \mathrm{Spec} k$. Let Λ be a finite field of characteristic $\ell \neq p$. Let X be a smooth scheme over S and $f : X \to Y$ a flat morphism of finite type to a smooth curve Y over S. If f has an isolated singularity at a closed point $x_0 \in |X|$, there is an invariant $\mu(X/Y, x_0)$ supported on x_0, called the Milnor number. The Milnor formula [4, Théorème 2.4] proved by Deligne says that the Milnor number is related to the total dimension at x_0 of the vanishing cycles $R\Phi(f, \Lambda)$ of f for the constant sheaf Λ, i.e.,

$$(-1)^n \mu(X/Y, x_0) = -\mathrm{dimtot} R\Phi_{\overline{x}_0}(f, \Lambda), \quad (1.1.1)$$

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where $n = \dim X$ and $\mathrm{dimtot} = \dim + \mathrm{Sw}$ denotes the total dimension. Later in [5], Deligne proposed a Milnor formula for any constructible sheaf $\mathcal{F}$ of Λ-modules on X, which is realized and proved by Saito in [11]. If $x_0 \in |X|$ is at most an isolated characteristic point of f with respect to the singular support of $\mathcal{F}$, then Saito's theorem [11, Theorem 5.9] says

$$(CC(\mathcal{F}), df)_{T^*X,x_0} = -\mathrm{dimtot} R\Phi_{\overline{x}_0}(f,\mathcal{F}), \tag{1.1.2}$$

where $CC(\mathcal{F})$ is the characteristic cycle of $\mathcal{F}$. Now we propose the following question:

Question 1.2. Is there a Milnor type formula for non-isolated singular/characteristic points?

1.3. If f is a projective flat morphism and if f is smooth outside $f^{-1}(y)$ for a closed point y of the curve Y, then the conductor formula of Bloch (cf. [12, Theorem 2.2.3 and Corollary 2.2.4])

$$-a_y(Rf_*\Lambda) = (-1)^n(X,X)_{T^*X,X_y} = (-1)^n \mathrm{deg} c^X_{n,X_y}(\Omega^1_{X/Y}) \cap [X] \tag{1.3.1}$$

gives a partial answer to the Question 1.2. We view (1.1.1), (1.1.2) and (1.3.1) in the form

$$\mathrm{deg}(\text{geometric class on singular locus}) = \mathrm{deg}(\text{cohomology class on singular locus}). \tag{1.3.2}$$

We expect the equality holds without taking degree, i.e.,

$$\mathrm{cl}(\text{geometric class}) = \text{cohomology class}, \tag{1.3.3}$$

where cl is the cycle class map. In the paper [15], the second author with Yigeng Zhao introduce a (cohomological) non-acyclicity class which is supported on non-acyclicity locus. Let Y be a smooth scheme over k and $X \to Y$ a separated morphism between schemes of finite type over k. Let $Z \subseteq X$ be a closed subscheme and $\mathcal{F} \in D_{\mathrm{ctf}}(X,\Lambda)$ such that $X \backslash Z \to Y$ is universally locally acyclic relatively to $\mathcal{F}|_{X\backslash Z}$. Then the cohomological non-acyclicity class $\widetilde{C}^Z_{X/Y/k}(\mathcal{F})$ is a class in $H^0_Z(X, \mathcal{K}_{X/Y/k})$, where $\mathcal{K}_{X/Y/k}$ sits in a distinguished triangle

$$\mathcal{K}_{X/Y} \to \mathcal{K}_{X/k} \to \mathcal{K}_{X/Y/k} \xrightarrow{+1}. \tag{1.3.4}$$

In this paper, we construct the geometric counterpart of the non-acyclicity class $\widetilde{C}^Z_{X/Y/k}(\mathcal{F})$. More precisely, when $X \to Y$ is a morphism between smooth schemes over k such that $X \to Y$ is $SS(\mathcal{F})$-transversal outside Z, then we construct a class $cc^Z_{X/Y/k}(\mathcal{F}) \in \mathrm{CH}_0(Z)$ (cf. (3.10.5)), called the geometric non-acyclicity class of $\mathcal{F}$. If moreover $\dim Z < \dim Y$, then we have the following fibration formula (3.10.5)

$$cc_{X/k}(\mathcal{F}) = c_{\mathrm{dimY}}(f^*\Omega^{1,\vee}_{Y/k}) \cap cc_{X/Y}(\mathcal{F}) + cc^Z_{X/Y/k}(\mathcal{F}). \tag{1.3.5}$$

Under certain conditions, we prove that the formation of the geometric non-acyclicity class $cc^Z_{X/Y/k}(\mathcal{F})$ is compatible with pullback (3.18.2) and proper push-forward (3.20.1). It also satisfies the Milnor formula (3.12.1) and a conductor formula (3.21.1). It is natural to expect the following conjecture holds:

Conjecture 1.4 (Conjecture 3.13). *We have*

$$\widetilde{C}^Z_{X/Y/k}(\mathcal{F}) = \widetilde{\mathrm{cl}}(cc^Z_{X/Y/k}(\mathcal{F})) \quad \text{in} \quad H^0_Z(X, \mathcal{K}_{X/Y/k}), \tag{1.4.1}$$

where $\widetilde{\mathrm{cl}} : \mathrm{CH}_0(Z) \xrightarrow{\mathrm{cl}} H^0_Z(X, \mathcal{K}_{X/k}) \xrightarrow{(1.3.4)} H^0_Z(X, \mathcal{K}_{X/Y/k})$ *and* cl *is the cycle class map.*

We hope (1.4.1) gives a formulation of Question 1.2 in some sense.

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Notation and Conventions.

(1) Let S be a Noetherian scheme and Sch_S the category of separated schemes of finite type over S.
(2) Let Λ be a Noetherian ring such that $m\Lambda = 0$ for some integer m invertible on S unless otherwise stated explicitly.
(3) For any scheme $X \in \mathrm{Sch}_S$, we denote by $D_{\mathrm{ctf}}(X, \Lambda)$ the derived category of complexes of Λ-modules of finite tor-dimension with constructible cohomology groups on X.
(4) For any separated morphism $f : X \to Y$ in Sch_S, we use the following notation

$$\mathcal{K}_{X/Y} = Rf^!\Lambda, \quad D_{X/Y}(-) = R\mathcal{H}om(-, \mathcal{K}_{X/Y}).$$

(5) For $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$ and $\mathcal{G} \in D_{\mathrm{ctf}}(Y, \Lambda)$ on S-schemes X and Y respectively, $\mathcal{F} \boxtimes^L_S \mathcal{G}$ denotes $\mathrm{pr}_1^*\mathcal{F} \otimes^L \mathrm{pr}_2^*\mathcal{G}$ on $X \times_S Y$.
(6) To simplify our notation, we omit to write R or L to denote the derived functors unless otherwise stated explicitly or for $R\mathcal{H}om$.

2. Transversality condition

2.1. We recall the (cohomological) transversality condition introduced in [15, 2.1], which is a relative version of the transversality condition studied by Saito [11, Definition 8.5]. Consider the following cartesian diagram in Sch_S:

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ {\scriptstyle p}\downarrow & \square & \downarrow{\scriptstyle f} \\ W & \xrightarrow{\delta} & T. \end{array} \tag{2.1.1}$$

Let $\mathcal{F} \in D_{\mathrm{ctf}}(Y, \Lambda)$ and $\mathcal{G} \in D_{\mathrm{ctf}}(T, \Lambda)$. Let $c_{\delta,f,\mathcal{F},\mathcal{G}}$ be the composition

$$\begin{aligned} c_{\delta,f,\mathcal{F},\mathcal{G}} : i^*\mathcal{F} \otimes^L p^*\delta^!\mathcal{G} &\xrightarrow{id\otimes \mathrm{b.c}} i^*\mathcal{F} \otimes^L i^!f^*\mathcal{G} \\ &\xrightarrow{\mathrm{adj}} i^!i_!(i^*\mathcal{F} \otimes^L i^!f^*\mathcal{G}) \\ &\xrightarrow[\simeq]{\mathrm{proj.formula}} i^!(\mathcal{F} \otimes^L i_!i^!f^*\mathcal{G}) \xrightarrow{\mathrm{adj}} i^!(\mathcal{F} \otimes^L f^*\mathcal{G}). \end{aligned} \tag{2.1.2}$$

We put $c_{\delta,f,\mathcal{F}} := c_{\delta,f,\mathcal{F},\Lambda} : i^*\mathcal{F} \otimes^L p^*\delta^!\Lambda \to i^!\mathcal{F}$. If $c_{\delta,f,\mathcal{F}}$ is an isomorphism, then we say that the morphism δ is *$\mathcal{F}$-transversal*. If $c_{i,\mathrm{id},\mathcal{F}}$ is an isomorphism, then we say i is $\mathcal{F}$-transversal.

By [15, 2.11], there is a functor $\delta^\Delta : D_{\mathrm{ctf}}(Y, \Lambda) \to D_{\mathrm{ctf}}(X, \Lambda)$ such that for any $\mathcal{F} \in D_{\mathrm{ctf}}(Y, \Lambda)$, we have a distinguished triangle

$$i^*\mathcal{F} \otimes^L p^*\delta^!\Lambda \xrightarrow{c_{\delta,f,\mathcal{F}}} i^!\mathcal{F} \to \delta^\Delta\mathcal{F} \xrightarrow{+1} . \tag{2.1.3}$$

Then δ is $\mathcal{F}$-transversal if and only if $\delta^\Delta(\mathcal{F})$=0 (cf. [15, Lemma 2.12]).

The following lemma gives an equivalence characterization between transversality condition and (universally) locally acyclicity condition.

Lemma 2.2. *Let $f : X \to S$ be a morphism of finite type between Noetherian schemes and $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$. The following conditions are equivalent:*

(1) *The morphism f is locally acyclic relatively to $\mathcal{F}$.*
(2) *The morphism f is universally locally acyclic relatively to $\mathcal{F}$.*
(3) *For any $\mathcal{G} \in D_{\mathrm{ctf}}(X, \Lambda)$, the canonical map*

$$D_{X/S}(\mathcal{G}) \boxtimes^L \mathcal{F} \to R\mathcal{H}om_{X\times_S X}(\mathrm{pr}_1^*\mathcal{G}, \mathrm{pr}_2^!\mathcal{F}) \tag{2.2.1}$$

is an isomorphism in $D_{\mathrm{ctf}}(X\times_S X,\Lambda)$*, where* $\mathrm{pr}_1: X\times_S X\to X$ *and* $\mathrm{pr}_2: X\times_S X\to X$ *are the projections.*

(4) *The canonical map*

$$D_{X/S}(\mathcal{F})\boxtimes^L\mathcal{F}\to R\mathcal{H}om_{X\times_S X}(\mathrm{pr}_1^*\mathcal{F},\mathrm{pr}_2^!\mathcal{F}) \tag{2.2.2}$$

is an isomorphism.

(5) *For any cartesian diagram between Noetherian schemes*

$$\begin{array}{ccc} Y\times_S X & \xrightarrow{\mathrm{pr}_2} & X \\ {\scriptstyle \mathrm{pr}_1}\downarrow & \square & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[\delta]{} & S \end{array} \tag{2.2.3}$$

the morphism δ *is* $\mathcal{F}$*-transversal.*

(6) *For any cartesian diagram* (2.2.3) *and any* $\mathcal{G}\in D_{\mathrm{ctf}}(S,\Lambda)$*, the morphism* $c_{\delta,f,\mathcal{F},\mathcal{G}}$ *is an isomorphism.*

(7) *For any cartesian diagram between Noetherian schemes*

$$\begin{array}{ccccc} Y\times_S X & \xrightarrow{\mathrm{pr}_2} & X' & \longrightarrow & X \\ {\scriptstyle \mathrm{pr}_1}\downarrow & \square & \downarrow{\scriptstyle f'} & \square & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[\delta]{} & S' & \longrightarrow & S, \end{array} \tag{2.2.4}$$

the morphism δ *is* $\mathcal{F}|_{X'}$*-transversal.*

(8) *For any cartesian diagram* (2.2.4) *and any* $\mathcal{G}\in D_{\mathrm{ctf}}(S',\Lambda)$*, the morphism* $c_{\delta,f',\mathcal{F}|_{X'},\mathcal{G}}$ *is an isomorphism.*

When S is a scheme of finite type over a field k, then the equivalence between (2) and (7) follows from [15, Proposition 2.4.(2) and Proposition 2.5]. In this case, we may require Y and S' smooth over k in (7).

Proof. By a result of Gabber [9, Corollary 6.6], (1) and (2) are equivalent. The equivalence between (2),(3) and (4) follows from [10, Proposition 2.5, Lemma 2.14, Theorem 2.16]. By [15, Proposition 2.4.(2)], (2) implies (6). It is clear that (6) implies (5).

Now we show (5) implies (1). Since δ is $\mathcal{F}$-transversal, we have an isomorphism

$$\mathcal{K}_{Y/S}\boxtimes^L\mathcal{F}\xrightarrow{\simeq}\mathrm{pr}_2^!\mathcal{F}. \tag{2.2.5}$$

By the projection formula, for any proper morphism $g: Y'\to Y$, the following canonical morphism

$$g_!g^!\mathcal{K}_{Y/S}\boxtimes^L\mathcal{F}\xrightarrow{\simeq}(g\times\mathrm{id})_!(g^!\mathcal{K}_{Y/S}\boxtimes^L\mathcal{F}) \tag{2.2.6}$$

is an isomorphism. If $g: Y'\to Y$ is an open immersion with closed complementary $\tau: Z=Y\backslash Y'\to Y$, we have a commutative diagram between distinguished triangles

$$\begin{array}{ccccccc} \tau_!\tau^!\mathcal{K}_{Y/S}\boxtimes^L\mathcal{F} & \longrightarrow & \mathcal{K}_{Y/S}\boxtimes^L\mathcal{F} & \longrightarrow & g_*g^*\mathcal{K}_{Y/S}\boxtimes^L\mathcal{F} & \xrightarrow{+1} & \\ {\scriptstyle (2.2.6)}\downarrow\simeq & & \downarrow\simeq & & \downarrow & & \\ (\tau\times\mathrm{id})_!(\tau^!\mathcal{K}_{Y/S}\boxtimes^L\mathcal{F}) & \longrightarrow & \mathcal{K}_{Y/S}\boxtimes^L\mathcal{F} & \longrightarrow & (g\times\mathrm{id})_*(g^*\mathcal{K}_{Y/S}\boxtimes^L\mathcal{F}) & & \\ {\scriptstyle (2.2.5)}\downarrow\simeq & & \| & & \downarrow\simeq & & \\ (\tau\times\mathrm{id})_!(\tau\times\mathrm{id})^!(\mathcal{K}_{Y/S}\boxtimes^L\mathcal{F}) & \longrightarrow & \mathcal{K}_{Y/S}\boxtimes^L\mathcal{F} & \longrightarrow & (g\times\mathrm{id})_*(g\times\mathrm{id})^*(\mathcal{K}_{Y/S}\boxtimes^L\mathcal{F}) & \xrightarrow{+1} & . \end{array} \tag{2.2.7}$$

Thus

$$g_*g^*\mathcal{K}_{Y/S}\boxtimes^L\mathcal{F}\to(g\times\mathrm{id})_*(g^*\mathcal{K}_{Y/S}\boxtimes^L\mathcal{F}) \tag{2.2.8}$$

is an isomorphism. Now we show that for any $\mathcal{H}\in D_{\mathrm{ctf}}(Y,\Lambda)$, the canonical morphism

$$D_{Y/S}(\mathcal{H})\boxtimes^L\mathcal{F}\to R\mathcal{H}om(\mathrm{pr}_1^*\mathcal{H},\mathrm{pr}_2^!\mathcal{F}) \tag{2.2.9}$$

is an isomorphism. We may assume $\mathcal{H}=j_!\Lambda$ for $j:U\to Y$ étale with U affine. Then (2.2.9) is equivalent to the following composition

$$j_*j^*\mathcal{K}_{Y/S}\boxtimes^L\mathcal{F}\to(j\times\mathrm{id})_*(j^*\mathcal{K}_{Y/S}\boxtimes^L\mathcal{F})\xrightarrow[\simeq]{(2.2.5)}(j\times\mathrm{id})_*(j\times\mathrm{id})^*\mathrm{pr}_2^!\mathcal{F}. \tag{2.2.10}$$

We show $j_*j^*\mathcal{K}_{Y/S}\boxtimes\mathcal{F}\to(j\times\mathrm{id})_*(j^*\mathcal{K}_{Y/S}\boxtimes\mathcal{F})$ is an isomorphism. We write j as a composition of a proper morphism and an open immersion. Then we may further assume that j is an open immersion. This is okay by (2.2.8). Thus (2.2.9) is an isomorphism. By [10, Theorem 2.16], the morphism $f:X\to S$ is (universally) locally acyclic relatively to $\mathcal{F}$.

We show (2) implies (8). By assumption, f' is also universally locally acyclic relatively to $\mathcal{F}|_{X'}$. Thus by (6), the morphism $c_{\delta,f',\mathcal{F}|_{X'},\mathcal{G}}$ is an isomorphism.

Finally, it is clear that (8) implies (7) and (7) implies (5). □

2.3. Now we recall the geometric transversality condition (cf. [1, 1.2] and [11, Definition 7.1 and Definition 5.3]). Let X be a smooth scheme over a field k. Let C be a conical closed subset of T^*X, i.e., a closed subset which is stable under the action of the multiplicative group $\mathbb{G}_m$. We denote by $T^*_XX\subseteq T^*X$ the zero section of the cotangent bundle T^*X of X.

(1) Let $h\colon W\to X$ be a morphism from a smooth scheme W over k. We say that h is C-transversal if the fiber $(C\times_XW)\cap dh^{-1}(T^*_WW)$ is contained in the zero-section $T^*_XX\times_XW\subseteq T^*X\times_XW$, where $dh\colon T^*X\times_XW\to T^*W$ is the canonical map.
(2) Assume that X and C are purely of dimension d and that W is purely of dimension m. We say that a C-transversal map $h\colon W\to X$ is properly C-transversal if every irreducible component of $C\times_XW$ is of dimension m.
(3) We say that a morphism $f\colon X\to Y$ to a smooth scheme Y over k is C-transversal if the inverse image $df^{-1}(C)$ is contained in the zero-section $T^*_YY\times_YX\subseteq T^*Y\times_YX$, where $df\colon T^*Y\times_YX\to T^*X$ is the canonical map.

The cohomological transversality condition and geometric transversality condition are related as follows. Let X be a smooth scheme of purely dimension d and Λ a finite local ring such that the characteristic ℓ of the residue field of Λ is invertible in k. Let $\mathcal{F}\in D_{\mathrm{ctf}}(X,\Lambda)$. The singular support $SS(\mathcal{F})$ defined by Beilinson [1] is a d-dimensional conical closed subset of T^*X. We have the following properties:

(1) $SS(\mathcal{F})\cap T^*_XX=\mathrm{supp}(\mathcal{F})$.
(2) Let $f:X\to Y$ be a morphism to a smooth scheme Y over k. If f is $SS(\mathcal{F})$-transversal, then f is universally locally acyclic relatively to $\mathcal{F}$.
(3) Let $f:W\to X$ be a morphism from a smooth scheme W over k. If f is $SS(\mathcal{F})$-transversal, then f is $\mathcal{F}$-transversal.

Now assume k is a perfect field. By [11, Theorem 5.9 and Theorem 5.19], the characteristic cycle $CC(\mathcal{F})$ is the unique d-cycle $CC(\mathcal{F})$ supported on $SS(\mathcal{F})$ with $\mathbb{Z}$-coefficients such that $CC(\mathcal{F})$ satisfies the Milnor formula (1.1.2) for $\mathcal{F}$.

3. Geometric non-acyclicity classes

3.1. Let S be a Noetherian scheme and Λ a Noetherian ring such that $m\Lambda = 0$ for some integer m invertible on S. Consider the following cartesian diagram in Sch_S

$$\begin{array}{ccc} X\times_S Y & \xrightarrow{\mathrm{pr}_1} & X \\ {\scriptstyle \mathrm{pr}_2}\downarrow & \square & \downarrow{\scriptstyle h} \\ Y & \xrightarrow{g} & S, \end{array} \tag{3.1.1}$$

where pr_1 and pr_2 are the projections. For any $\mathcal{F}\in D_{\mathrm{ctf}}(X,\Lambda)$ and $\mathcal{G}\in D_{\mathrm{ctf}}(Y,\Lambda)$, we have canonical morphisms

$$\mathcal{F}\boxtimes^L_S \mathcal{K}_{Y/S} = \mathrm{pr}_1^*\mathcal{F}\otimes^L \mathrm{pr}_2^* g^!\Lambda \xrightarrow{c_{g,h,\mathcal{F}}} \mathrm{pr}_1^!\mathcal{F}, \tag{3.1.2}$$

$$\mathcal{F}\boxtimes^L_S D_{Y/S}(\mathcal{G}) \to R\mathcal{H}om_{X\times_S Y}(\mathrm{pr}_2^*\mathcal{G}, \mathrm{pr}_1^!\mathcal{F}), \tag{3.1.3}$$

where (3.1.3) is adjoint to

$$\mathcal{F}\boxtimes^L_S (D_{Y/S}(\mathcal{G})\otimes^L \mathcal{G}) \xrightarrow{id\boxtimes \mathrm{ev}} \mathcal{F}\boxtimes^L_S \mathcal{K}_{Y/S} \xrightarrow{(3.1.2)} \mathrm{pr}_1^!\mathcal{F}. \tag{3.1.4}$$

Note that (3.1.2) is a special case of (3.1.3) by taking $\mathcal{G}=\Lambda$. If moreover $X\to S$ is universally locally acyclic relatively to $\mathcal{F}$, then (3.1.3) is an isomorphism by (2.2.9). For a morphism $c = (c_1, c_2): C\to X\times_S Y$, we have a canonical isomorphism by [3, Corollaire 3.1.12.2]

$$R\mathcal{H}om(c_2^*\mathcal{G}, c_1^!\mathcal{F}) \xrightarrow{\simeq} c^! R\mathcal{H}om(\mathrm{pr}_2^*\mathcal{G}, \mathrm{pr}_1^!\mathcal{F}). \tag{3.1.5}$$

3.2. Consider a commutative diagram in Sch_S:

$$\begin{array}{ccccc} Z & \overset{\tau}{\hookrightarrow} & X & \xrightarrow{f} & Y, \\ & & & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle g} & \\ & & & S & \end{array} \tag{3.2.1}$$

where $\tau: Z\to X$ is a closed immersion and g is a smooth morphism. Let $i: X\times_Y X\to X\times_S X$ be the base change of the diagonal morphism $\delta: Y\to Y\times_S Y$:

$$\begin{array}{ccc} X & = & X \\ {\scriptstyle \delta_1}\downarrow & \square & \downarrow{\scriptstyle \delta_0} \\ X\times_Y X & \xrightarrow{i} & X\times_S X \\ {\scriptstyle p}\downarrow & \square & \downarrow{\scriptstyle f\times f} \\ Y & \xrightarrow{\delta} & Y\times_S Y, \end{array} \quad (f: X\to Y) \tag{3.2.2}$$

where δ_0 and δ_1 are the diagonal morphisms. Put $\mathcal{K}_{X/Y/S} := \delta^\Delta\mathcal{K}_{X/S}\simeq \delta_1^*\delta^\Delta\delta_{0*}\mathcal{K}_{X/S}$. By (2.1.3), we have the following distinguished triangle (see also [15, (4.2.5)])

$$\mathcal{K}_{X/Y}\to\mathcal{K}_{X/S}\to\mathcal{K}_{X/Y/S}\xrightarrow{+1}. \tag{3.2.3}$$

Let $\mathcal{F}\in D_{\mathrm{ctf}}(X,\Lambda)$ such that $X\backslash Z\to Y$ is universally locally acyclic relatively to $\mathcal{F}|_{X\backslash Z}$ and that $h: X\to S$ is universally locally acyclic relatively to $\mathcal{F}$. We put

$$\mathcal{H}_S = R\mathcal{H}om_{X\times_S X}(\mathrm{pr}_2^*\mathcal{F}, \mathrm{pr}_1^!\mathcal{F}), \qquad \mathcal{T}_S = \mathcal{F}\boxtimes^L_S D_{X/S}(\mathcal{F}). \tag{3.2.4}$$

The relative cohomological characteristic class $C_{X/S}(\mathcal{F})$ is the composition (cf. [15, 3.1])

$$\Lambda \xrightarrow{\text{id}} R\mathcal{H}om(\mathcal{F},\mathcal{F}) \xrightarrow[\simeq]{(3.1.5)} \delta_0^!\mathcal{H}_S \xleftarrow[\simeq]{(3.1.3)} \delta_0^!\mathcal{T}_S \to \delta_0^*\mathcal{T}_S \xrightarrow{\text{ev}} \mathcal{K}_{X/S}. \tag{3.2.5}$$

By the assumption on $\mathcal{F}$, $\delta_1^*\delta^\Delta\mathcal{T}_S$ is supported on Z by [15, 4.4]. The non-acyclicity class $\widetilde{C}^Z_{X/Y/S}(\mathcal{F})$ is the composition (cf. [15, Definition 4.6])

$$\Lambda \to \delta_0^!\mathcal{H}_S \xleftarrow{\simeq} \delta_0^!\mathcal{T}_S \simeq \delta_1^! i^!\mathcal{T}_S \to \delta_1^* i^!\mathcal{T}_S \to \delta_1^*\delta^\Delta\mathcal{T}_S \xleftarrow{\simeq} \tau_*\tau^!\delta_1^*\delta^\Delta\mathcal{T}_S \to \tau_*\tau^!\mathcal{K}_{X/Y/S}. \tag{3.2.6}$$

If the following condition holds:

$$H^0(Z,\mathcal{K}_{Z/Y}) = 0 \text{ and } H^1(Z,\mathcal{K}_{Z/Y}) = 0 \tag{3.2.7}$$

then the map $H^0_Z(X,\mathcal{K}_{X/S}) \xrightarrow{(3.2.3)} H^0_Z(X,\mathcal{K}_{X/Y/S})$ is an isomorphism. In this case, the class $\widetilde{C}^Z_{X/Y/S}(\mathcal{F}) \in H^0_Z(X,\mathcal{K}_{X/Y/S})$ defines an element of $H^0_Z(X,\mathcal{K}_{X/S})$, which is denoted by $C^Z_{X/Y/S}(\mathcal{F})$. Now we summarize the functorial properties for the non-acyclicity classes (cf. [15, Theorem 1.9, Proposition 1.11, Theorem 1.12, Theorem 1.14]).

Proposition 3.3. *Let us denote the diagram (3.2.1) simply by $\Delta = \Delta^Z_{X/Y/S}$ and $\widetilde{C}^Z_{X/Y/S}(\mathcal{F})$ by $C_\Delta(\mathcal{F})$. Let $\mathcal{F} \in D_{\mathrm{ctf}}(X,\Lambda)$. Assume that $Y \to S$ is smooth, $X\backslash Z \to Y$ is universally locally acyclic relatively to $\mathcal{F}|_{X\backslash Z}$ and that $X \to S$ is universally locally acyclic relatively to $\mathcal{F}$.*

(1) (*Fibration formula*) *If $H^0(Z,\mathcal{K}_{Z/Y}) = H^1(Z,\mathcal{K}_{Z/Y}) = 0$, then we have*

$$C_{X/S}(\mathcal{F}) = c_r(f^*\Omega^{1,\vee}_{Y/S}) \cap C_{X/Y}(\mathcal{F}) + C_\Delta(\mathcal{F}) \quad \text{in} \quad H^0(X,\mathcal{K}_{X/S}). \tag{3.3.1}$$

(2) (*Pull-back*) *Let $b: S' \to S$ be a morphism of Noetherian schemes. Let $\Delta' = \Delta^{Z'}_{X'/Y'/S'}$ be the base change of $\Delta = \Delta^Z_{X/Y/S}$ by $b: S' \to S$. Let $b_X: X' = X\times_S S' \to X$ be the base change of b by $X \to S$. Then we have*

$$b_X^* C_\Delta(\mathcal{F}) = C_{\Delta'}(b_X^*\mathcal{F}) \quad \text{in} \quad H^0_{Z'}(X',\mathcal{K}_{X'/Y'/S'}), \tag{3.3.2}$$

where $b_X^*: H^0_Z(X,\mathcal{K}_{X/Y/S}) \to H^0_{Z'}(X',\mathcal{K}_{X'/Y'/S'})$ is the induced pull-back morphism.

(3) (*Proper push-forward*) *Consider a diagram $\Delta' = \Delta^{Z'}_{X'/Y/S}$. Let $s: X \to X'$ be a proper morphism over Y such that $Z \subseteq s^{-1}(Z')$. Then we have*

$$s_*(C_\Delta(\mathcal{F})) = C_{\Delta'}(Rs_*\mathcal{F}) \quad \text{in} \quad H^0_{Z'}(X',\mathcal{K}_{X'/Y/S}), \tag{3.3.3}$$

where $s_*: H^0_Z(X,\mathcal{K}_{X/Y/S}) \to H^0_{Z'}(X',\mathcal{K}_{X'/Y/S})$ is the induced push-forward morphism.

(4) (*Cohomological Milnor formula*) *Assume $S = \mathrm{Spec}k$ for a perfect field k of characteristic $p > 0$ and Λ is a finite local ring such that the characteristic of the residue field is invertible in k. If $Z = \{x\}$, then we have*

$$C_\Delta(\mathcal{F}) = -\mathrm{dimtot}R\Phi_{\bar{x}}(\mathcal{F},f) \quad \text{in} \quad \Lambda = H^0_x(X,\mathcal{K}_{X/k}), \tag{3.3.4}$$

where $R\Phi(\mathcal{F},f)$ is the complex of vanishing cycles and $\mathrm{dimtot} = \dim + \mathrm{Sw}$ is the total dimension.

(5) (*Cohomological conductor formula*) *Assume $S = \mathrm{Spec}k$ for a perfect field k of characteristic $p > 0$ and Λ is a finite local ring such that the characteristic of the residue field is invertible in k. If Y is a smooth connected curve over k and $Z = f^{-1}(y)$ for a closed point $y \in |Y|$, then we have*

$$f_*C_\Delta(\mathcal{F}) = -a_y(Rf_*\mathcal{F}) \quad \text{in} \quad \Lambda = H^0_y(Y,\mathcal{K}_{Y/k}), \tag{3.3.5}$$

where $a_y(\mathcal{G}) = \mathrm{rank}\mathcal{G}|_{\bar{\eta}} - \mathrm{rank}\mathcal{G}_{\bar{y}} + \mathrm{Sw}_y\mathcal{G}$ is the Artin conductor of an object $\mathcal{G} \in D_{\mathrm{ctf}}(Y,\Lambda)$ at y and η is the generic point of Y.

The formation of non-acyclicity classes is also compatible with specialization maps (cf. [15, Proposition 4.17]).

3.4. Let X be a smooth connected curve over k. Let $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$ and $Z \subseteq X$ be a finite set of closed points such that the cohomology sheaves of $\mathcal{F}|_{X \backslash Z}$ are locally constant. By the cohomological Milnor formula (3.3.4), we have the following (motivic) expression for the Artin conductor of $\mathcal{F}$ at $x \in Z$

$$a_x(\mathcal{F}) = \mathrm{dimtot} R\Phi_{\bar{x}}(\mathcal{F}, \mathrm{id}) = -C^{\{x\}}_{U/U/k}(\mathcal{F}|_U), \tag{3.4.1}$$

where U is any open subscheme of X such that $U \cap Z = \{x\}$. By (3.3.1), we get the following cohomological Grothendieck-Ogg-Shafarevich formula (cf. [15, Corollary 6.6]):

$$C_{X/k}(\mathcal{F}) = \mathrm{rank}\mathcal{F} \cdot c_1(\Omega^{1,\vee}_{X/k}) - \sum_{x \in Z} a_x(\mathcal{F}) \cdot [x] \quad \text{in} \quad H^0(X, \mathcal{K}_{X/k}). \tag{3.4.2}$$

Here we give a new proof of (3.4.1) by using Gabber-Katz extension (cf. [8]). For simplicity, we assume $Z = \{x\}$ and k is algebraically closed. Since the formation $C^{\{x\}}_{X/X/k}(\mathcal{F})$ is etale local around x, we may assume there is an etale morphism $f : X \to \mathbb{P}^1_k$ such that $f(x) = \infty$. Let $\mathcal{G}$ be the Gabber-Katz extension of $\mathcal{F}|_{X_{(\bar{x})}}$ to $\mathbb{G}_m$. Then $\mathcal{G}$ is smooth on $\mathbb{G}_m$, tamely ramified at $0 \in \mathbb{A}^1_k$ and $\mathcal{G}|_{X_{(\bar{x})}} \simeq \mathcal{F}|_{X_{(\bar{x})}}$. Let $A = \mathbb{P}^1_k \backslash \{0\}$. We have $C^{\{x\}}_{X/X/k}(\mathcal{F}) = C^{\{\infty\}}_{A/A/k}(\mathcal{G})$. By the formula (3.3.1) and the Grothendieck-Ogg-Shafarevich formula for $\mathbb{P}^1$, we get

$$-C^{\{\infty\}}_{A/A/k}(\mathcal{G}) - C^{\{0\}}_{\mathbb{A}^1_k/\mathbb{A}^1_k/k}(\mathcal{G}) = a_\infty(\mathcal{G}) + a_0(\mathcal{G}). \tag{3.4.3}$$

We only need to show: $-C^{\{0\}}_{\mathbb{A}^1_k/\mathbb{A}^1_k/k}(\mathcal{G}) = a_0(\mathcal{G})$. Replacing $\mathcal{G}$ by the Gabber-Katz extension of $\mathcal{G}|_{\mathbb{A}^1_{k,(\bar{0})}}$, we may assume $\mathcal{G}$ is a smooth sheaf on $\mathbb{G}_m$ such that $\mathcal{G}$ is tamely ramified at 0 and ∞. We may further assume $\mathcal{G}_{\bar{0}} = \mathcal{G}_{\bar{\infty}} = 0$. By the formula (3.3.1) and the Grothendieck-Ogg-Shafarevich formula for $\mathbb{P}^1$, we get

$$-2C^{\{0\}}_{\mathbb{A}^1_k/\mathbb{A}^1_k/k}(\mathcal{G}) = 2a_0(\mathcal{G}) = 2\mathrm{rank}\mathcal{G}, \tag{3.4.4}$$

which implies $-C^{\{0\}}_{\mathbb{A}^1_k/\mathbb{A}^1_k/k}(\mathcal{G}) = a_0(\mathcal{G}) = \mathrm{rank}\mathcal{G}$. This finishes the proof of (3.4.2).

3.5. Now we start to construct and prove a geometric counterpart of Proposition 3.3. Let k be a perfect field of characteristic p and Λ be a finite local ring whose residue field is of characteristic $\ell \neq p$. Let Sm_k be the category of smooth schemes over k. Let S be a smooth connected scheme of dimension s over k. Let $f : X \to S$ be a morphism in Sm_k. Let $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$ such that f is $SS(\mathcal{F})$-transversal. Consider the following morphisms

$$X \xrightarrow{0} T^*S \times_S X \xrightarrow{df} T^*X, \tag{3.5.1}$$

where 0 stands for the zero section. By assumption, $df^{-1}(SS(\mathcal{F}))$ is contained in $0(X)$. We define the relative characteristic class of $\mathcal{F}$ to be the following s-cycle class on X:

$$cc_{X/S}(\mathcal{F}) := (-1)^s \cdot (df)^!(CC(\mathcal{F})) \quad \text{in} \quad \mathrm{CH}_s(X), \tag{3.5.2}$$

where $(df)^!$ is the refined Gysin pullback. We don't know how to define $cc_{X/S}(\mathcal{F})$ if one only assume f is universally locally acyclic relatively to $\mathcal{F}$. When f is a smooth morphism, then we have a cartesian diagram

$$\begin{array}{ccc} T^*S\times_S X & \xrightarrow{df} & T^*X \\ \downarrow & \square & \downarrow \\ X & \xrightarrow[0_{X/S}]{} & T^*(X/S). \end{array} \tag{3.5.3}$$

In this case, we have $cc_{X/S}(\mathcal{F}) = (-1)^s \cdot 0^!_{X/S}(CC(\mathcal{F}))$ (cf. [14, Definition 2.11]). If f is a smooth morphism of relative dimension r and if $\mathcal{F}$ is locally constant, then we have

$$cc_{X/S}(\mathcal{F}) = (-1)^s \cdot 0^!_{X/S}((-1)^{\dim X} \cdot \mathrm{rank}\mathcal{F} \cdot [X]) = \mathrm{rank}\mathcal{F} \cdot c_r(\Omega^{1,\vee}_{X/S}) \cap [X]. \tag{3.5.4}$$

We propose the following conjecture:

Conjecture 3.6. *Let S be a smooth connected scheme of dimension s over k. Let $f : X \to S$ be a morphism in Sm_k. Let $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$ such that f is $SS(\mathcal{F})$-transversal. Then we have*

$$\mathrm{cl}(cc_{X/S}(\mathcal{F})) = C_{X/S}(\mathcal{F}) \quad \text{in} \quad H^0(X, \mathcal{K}_{X/S}), \tag{3.6.1}$$

where $\mathrm{cl} : \mathrm{CH}_s(X) \to H^0(X, \mathcal{K}_{X/S})$ *is the cycle class map.*

When $S = \mathrm{Spec}k$, then it is Saito's conjecture [11, Conjecture 6.8.1], which is proved under quasi-projective assumption in [15, Theorem 1.3]. When $f : X \to S$ is a smooth morphism, then (3.6.1) is true for a locally constant constructible (flat) sheaf $\mathcal{F}$ of Λ-modules. Indeed, this follows from (3.5.4), [15, Lemma 3.3] and (3.3.1).

Question 3.7. How to define a relative cycle class map from groups of relative cycle classes to $H^0(X, \mathcal{K}_{X/S})$? It is interesting to see whether $cc_{X/S}(\mathcal{F})$ is a relative cycle class over S. Is there a canonical way to lift $cc_{X/S}(\mathcal{F})$ to a relative cycle (other than a class)?

3.8. Consider a commutative diagram in Sm_k:

$$\begin{array}{ccccc} Z & \overset{\tau}{\hookrightarrow} & X & \xrightarrow{f} & Y \\ & & & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle g} & \\ & & & S & \end{array} \tag{3.8.1}$$

where $\tau : Z \to X$ is a closed immersion, g is a smooth morphism of relative dimension r and $s = \dim S$. Let $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$ such that $X \backslash Z \to Y$ is $SS(\mathcal{F}|_{X\backslash Z})$-transversal and that $X \to S$ is $SS(\mathcal{F})$-transversal. We have a commutative diagram on vector bundles

$$\begin{array}{ccccc} X & = & X & & \\ \downarrow & \square & \downarrow{\scriptstyle 0} & & \\ T^*S\times_S X & \xrightarrow{dg_X} & T^*Y\times_Y X & \xrightarrow{df} & T^*X \\ \downarrow & \square & \downarrow & & \\ T^*S\times_S Y & \xrightarrow{dg} & T^*Y & & \\ \downarrow & \square & \downarrow & & \\ Y & \xrightarrow{0} & T^*(Y/S), & & \end{array} \tag{3.8.2}$$

where dg_X is the base change of dg. By assumption, $df^{-1}(SS(\mathcal{F}))$ is supported on $0(X) \cup T^*Y \times_Y Z$ and $dh^{-1}(SS(\mathcal{F})) = dg_X{}^{-1} df^{-1}(SS(\mathcal{F}))$ is contained in the zero section $0(X) \subseteq T^*S \times_S X$. Consider the following class on $df^{-1}(SS(\mathcal{F})) \cap (T^*Y \times_Y Z)$

$$df^!(CC(\mathcal{F}))|_{T^*Y \times_Y Z} := ((T^*Y \times_Y X) \cdot CC(\mathcal{F}))^{df^{-1}(SS(\mathcal{F})) \cap (T^*Y \times_Y Z)}, \tag{3.8.3}$$

which is the part of $df^!(CC(\mathcal{F}))$ supported on $df^{-1}(SS(\mathcal{F})) \cap (T^*Y \times_Y Z)$ (cf. [6, P.95]). We define the geometric non-acyclicity class $cc^Z_{X/Y/S}(\mathcal{F})$ of $\mathcal{F}$ to be

$$cc^Z_{X/Y/S}(\mathcal{F}) := (-1)^s \cdot dg^!_X(df^!(CC(\mathcal{F}))|_{T^*Y \times_Y Z}) \quad \text{in} \quad \mathrm{CH}_s(Z). \tag{3.8.4}$$

Remark 3.9. If $Z = X$, then $cc^Z_{X/Y/S}(\mathcal{F}) = cc_{X/S}(\mathcal{F})$.

3.10. Assume moreover that $\dim Z < r + s$. Then the restriction map $\mathrm{CH}_{r+s}(X) \xrightarrow{\simeq} \mathrm{CH}_{r+s}(X \backslash Z)$ is an isomorphism. In this case, we define the relative characteristic class $cc_{X/Y}(\mathcal{F})$ to be

$$cc_{X/Y}(\mathcal{F}) := cc_{X \backslash Z/Y}(\mathcal{F}|_{X \backslash Z}) \quad \text{in} \quad \mathrm{CH}_{r+s}(X), \tag{3.10.1}$$

which is also equal to $(-1)^{r+s} \cdot ((T^*Y \times_Y X) \cdot CC(\mathcal{F}))^{0(X)}$, which is the part of $(-1)^{r+s} \cdot df^! CC(\mathcal{F})$ supported on $0(X)$. Then we have

$$(-1)^s \cdot df^!(CC(\mathcal{F})) = (-1)^r \cdot cc_{X/Y}(\mathcal{F}) + (-1)^s \cdot df^!(CC(\mathcal{F}))|_{T^*Y \times_Y Z}. \tag{3.10.2}$$

Applying $dg^!_X$ to the above formula, we get

$$cc_{X/S}(\mathcal{F}) = (-1)^r \cdot dg^!_X cc_{X/Y}(\mathcal{F}) + cc^Z_{X/Y/S}(\mathcal{F}) \quad \text{in} \quad \mathrm{CH}_s(X). \tag{3.10.3}$$

By the excess intersection formula [6, Theorem 6.3], we have

$$(-1)^r \cdot dg^!_X cc_{X/Y}(\mathcal{F}) = c_r(f^*\Omega^{1,\vee}_{Y/S}) \cap cc_{X/Y}(\mathcal{F}). \tag{3.10.4}$$

Thus if $\dim Z < r + s$, then we have

$$cc_{X/S}(\mathcal{F}) = c_r(f^*\Omega^{1,\vee}_{Y/S}) \cap cc_{X/Y}(\mathcal{F}) + cc^Z_{X/Y/S}(\mathcal{F}). \tag{3.10.5}$$

In particular, if Z is empty, then we have

$$cc_{X/S}(\mathcal{F}) = c_r(f^*\Omega^{1,\vee}_{Y/S}) \cap cc_{X/Y}(\mathcal{F}). \tag{3.10.6}$$

Remark 3.11. Assume that $X \to S$ is smooth of relative dimension r and that $X \backslash Z \to Y$ is smooth of relative dimension n ($n < r$). Then $\Omega^1_{X/Y}$ is locally free of rank n on $X \backslash Z$ and we have the localized Chern classes $c^X_{i,Z}(\Omega^1_{X/Y})$ for $i > n$ (cf. [2, Section 1]). By [12, Lemma 2.1.4], we have

$$cc^Z_{X/Y/S}(\Lambda) = (-1)^r c^X_{r,Z}(\Omega^1_{X/Y}) \cap [X] \quad \text{in} \quad \mathrm{CH}_s(Z). \tag{3.11.1}$$

Theorem 3.12 (Saito's Milnor formula)**.** *Assume $S = \mathrm{Spec}k$. Let X be a smooth scheme over S and $f : X \to Y = \mathbb{A}^1_k$ a separated morphism. Let x be a closed point of X and $Z = \{x\}$. Let $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$ such that f is $SS(\mathcal{F})$-transversal outside Z. Then we have*

$$cc^Z_{X/Y/S}(\mathcal{F}) = -\mathrm{dimtot} R\Phi_{\bar{x}}(\mathcal{F}, f) \quad \text{in} \quad \mathbb{Z} = \mathrm{CH}_0(\{x\}). \tag{3.12.1}$$

Proof. By [13, (3.4.5.4)-(3.4.5.5)], we have $cc^Z_{X/Y/S}(\mathcal{F}) = (CC(\mathcal{F}), df)_{T^*X,x} \cdot [x]$. Now the result follows from Saito's Milnor formula [11, Theorem 5.9]. □

We expect the following Milnor type formula for non-isolated singular/characteristic points holds.

Conjecture 3.13. *Let S be a smooth connected k-scheme of dimension s. Consider the commutative diagram (3.8.1). Let $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$ such that $X \backslash Z \to Y$ is $SS(\mathcal{F}|_{X\backslash Z})$-transversal and that $X \to S$ is $SS(\mathcal{F})$-transversal. Then we have an equality*

$$\widetilde{C}^{Z}_{X/Y/S}(\mathcal{F}) = \widetilde{\mathrm{cl}}(cc^{Z}_{X/Y/S}(\mathcal{F})) \quad \text{in} \quad H^0_Z(X, \mathcal{K}_{X/Y/S}), \tag{3.13.1}$$

where $\widetilde{\mathrm{cl}}$ is the composition $\mathrm{CH}_s(Z) \xrightarrow{\mathrm{cl}} H^0_Z(X, \mathcal{K}_{X/S}) \xrightarrow{(3.2.3)} H^0_Z(X, \mathcal{K}_{X/Y/S})$.

When $S = \mathrm{Spec}k$, $Y = \mathbb{A}^1_k$ and $Z = \{x\}$, then Conjecture 3.13 follows from Saito's Milnor formula (3.12.1) and the cohomological Milnor formula (3.3.4).

When $Z = X$, then $\widetilde{C}^{Z}_{X/Y/S}(\mathcal{F}) = C_{X/S}(\mathcal{F})$ in $H^0(X, \mathcal{K}_{X/Y/S})$ and $cc^{Z}_{X/Y/S}(\mathcal{F}) = cc_{X/S}(\mathcal{F})$ in $\mathrm{CH}_s(X)$. In this case, (3.13.1) is a weak version of Conjecture 3.6.

Remark 3.14. Let $f : X \to Y$ be a separated morphism between smooth schemes over k. Let $Z \subseteq X$ be a closed subset and $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$. Assume that f is universally locally acyclicity outside Z. Let $n = \dim X$. We expect that there is a n-cycle $CC^{Z}_{X/Y}(\mathcal{F})$ supported on $T^*X \times_X Z$ such that

$$\mathrm{cl}(0^!_X CC^{Z}_{X/Y}(\mathcal{F})) = \widetilde{C}^{Z}_{X/Y/k}(\mathcal{F}) \quad \text{in} \quad H^0_Z(X, \mathcal{K}_{X/Y/k}). \tag{3.14.1}$$

If Y is a smooth curve and Z is a finite set of closed points of X, then

$$CC^{Z}_{X/Y}(\mathcal{F}) = -\sum_{x \in Z} \mathrm{dimtot} R\Phi_{\bar{x}}(\mathcal{F}, f) \cdot [T^*_x X]. \tag{3.14.2}$$

If $f = \mathrm{id}$ and Z is the smallest closed subset of X such that $\mathcal{F}|_{X \backslash Z}$ is smooth, then

$$CC^{Z}_{X/Y}(\mathcal{F}) = CC(\mathcal{F}) - \mathrm{rank}\mathcal{F} \cdot CC(\Lambda). \tag{3.14.3}$$

In order to construct $CC^{Z}_{X/Y}(\mathcal{F})$, we will introduce f-singular support (singular support with respect to a morphism $f : X \to Y$). When f is the identity morphism, then id-singular support is the singular support defined by Beilinson [1]. We expect the f-singular support is exist under suitable conditions and the non-acyclicity cycle $CC^{Z}_{X/Y}(\mathcal{F})$ is a cycle supported on the f-singular support. Details will appear in the near future.

Proposition 3.15. *Consider a cartesian diagram in Sm_k*

$$\begin{array}{ccc} X' & \xrightarrow{i} & X \\ {\scriptstyle f'}\downarrow & \square & \downarrow{\scriptstyle f} \\ S' & \xrightarrow[\delta]{} & S. \end{array} \tag{3.15.1}$$

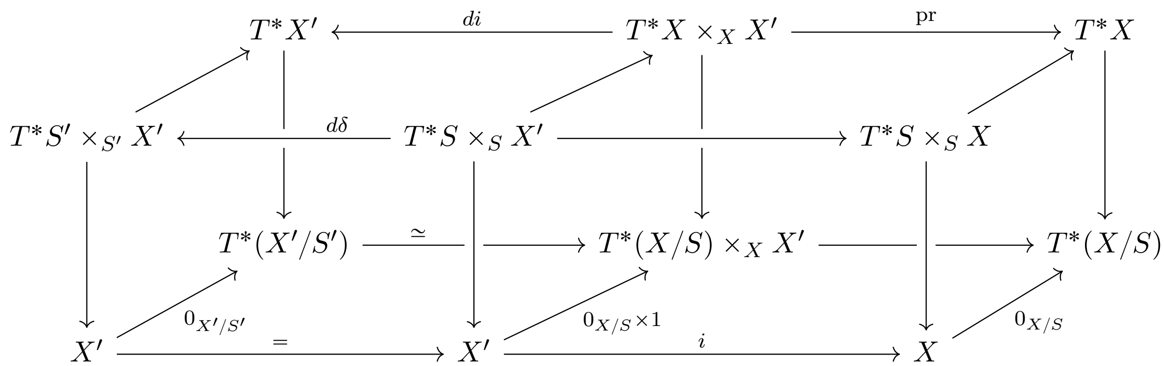
We assume that f and f' are smooth morphisms, and S and S' are connected of dimension s and s' respectively. Let $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$. Assume that $X \to S$ is $SS(\mathcal{F})$-transversal and i is properly $SS(\mathcal{F})$-transversal. Then we have

$$i^! cc_{X/S}(\mathcal{F}) = cc_{X'/S'}(i^*\mathcal{F}) \quad \text{in} \quad CH_{s'}(X'), \tag{3.15.2}$$

where $i^! : CH_s(X) \to CH_{s'}(X')$ is the refined Gysin pull-back.

Since f is $SS(\mathcal{F})$-transversal, the morphism i is $SS(\mathcal{F})$-transversal. We don't know how to remove the properly assumption on i.

Proof. We consider the following diagram



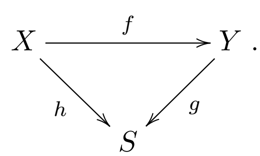
Note that the square containing the morphisms di and $d\delta$ is cartesian. In the following calculations, even though di and $d\delta$ are not proper, but we can still applying di_* and $d\delta_*$ since di is finite on the support of $\mathrm{pr}^{-1}(SS\mathcal{F})$ and $d\delta$ is finite on the zero section X' of $T^*S\times_S X'$. We have

$$
\begin{aligned}
cc_{X'/S'}(i^*\mathcal{F}) &= (-1)^{s'}\cdot 0^!_{X'/S'}CC(i^*\mathcal{F})\\
&\overset{(a)}{=} (-1)^{s'}\cdot 0^!_{X'/S'}(di_*\mathrm{pr}^!CC(\mathcal{F})\cdot(-1)^{-\dim(X')+\dim(X)})\\
&= d\delta_* 0^!_{X'/S'}\mathrm{pr}^!CC(\mathcal{F})\cdot(-1)^{s'-\dim(X')+\dim(X)}\\
&\overset{(b)}{=} 0^!_{X'/S'}\mathrm{pr}^!CC(\mathcal{F})\cdot(-1)^{s} = 0^!_{X/S}\mathrm{pr}^!CC(\mathcal{F})\cdot(-1)^{s}\\
&= i^!(0^!_{X/S}CC(\mathcal{F})\cdot(-1)^{s}) = i^! cc_{X/S}(\mathcal{F}).
\end{aligned}
\tag{3.15.3}
$$

where (a) follows from [11, Theorem 7.6] and (b) follows from the fact that $0^!_{X'/S'}\mathrm{pr}^!CC(\mathcal{F})$ is supported on the zero section of $T^*S\times_S X'$. □

3.16. Consider a commutative diagram in Sm_k:

(3.16.1)



Let $\mathcal{F}\in D_{\mathrm{ctf}}(X,\Lambda)$ such that h is $SS(\mathcal{F})$-transversal. Assume f is proper on $B(SS(\mathcal{F}))=\mathrm{supp}(\mathcal{F})$. By [11, Lemma 3.8], g is $f_\circ SS(\mathcal{F})$-transversal. By [1, Lemma 2.2(ii)], $SS(Rf_*\mathcal{F})\subseteq f_\circ SS(\mathcal{F})$. Thus $g: Y\to S$ is also $SS(Rf_*\mathcal{F})$-transversal and the class $cc_{Y/S}(Rf_*\mathcal{F})$ is well-defined.

Proposition 3.17. *Consider the assumptions in 3.16. Assume moreover that Y is projective, $f: X\to Y$ is quasi-projective and $\dim f_\circ SS(\mathcal{F})\leqslant \dim Y$. Then we have*

$$f_* cc_{X/S}(\mathcal{F}) = cc_{Y/S}(Rf_*\mathcal{F}) \quad \text{in} \quad \mathrm{CH}_s(Y).$$

We don't know how to remove the assumption $\dim f_\circ SS(\mathcal{F})\leqslant\dim Y$.

Proof. Consider the following commutative diagram

$$\begin{array}{ccccc} & & X & \xrightarrow{f} & Y \\ & & \downarrow{\scriptstyle 0_X} & & \downarrow{\scriptstyle 0_Y} \\ & & T^*S\times_S X & \xrightarrow{\mathrm{id}\times f} & T^*S\times_S Y \\ & \swarrow{\scriptstyle dh} & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle dg} \\ T^*X & \xleftarrow[df]{} & T^*Y\times_Y X & \xrightarrow{\mathrm{id}\times f} & T^*Y \end{array} \tag{3.17.1}$$

Then we have

$$\begin{aligned} f_*cc_{X/S}(\mathcal{F}) &\overset{(3.5.2)}{=} (-1)^s\cdot f_*dh^!(CC(\mathcal{F})) = (-1)^s\cdot(\mathrm{id}\times f)_*r^!df^!(CC(\mathcal{F})) \\ &= (-1)^s\cdot dg^!(\mathrm{id}\times f)_*df^!(CC(\mathcal{F})) \\ &\overset{(a)}{=} (-1)^s\cdot dg^!CC(Rf_*\mathcal{F}) \overset{(3.5.2)}{=} cc_{Y/S}(Rf_*\mathcal{F}). \end{aligned} \tag{3.17.2}$$

where (a) follows from [12, Theorem 2.2.5]. □

Proposition 3.18. *Consider a commutative diagram in* Sm_k

$$\begin{array}{ccccc} X' & & \xrightarrow{i_X} & X & \\ \downarrow{\scriptstyle h'} & \searrow{\scriptstyle f'} & & \downarrow{\scriptstyle h} & \searrow{\scriptstyle f} \\ & Y' & \xrightarrow[i_Y]{} & & Y \\ & \swarrow{\scriptstyle g'} & & & \swarrow{\scriptstyle g} \\ S' & & \xrightarrow[\delta]{} & S, & \end{array} \tag{3.18.1}$$

where squares are cartesian diagrams. Let $Z\subseteq X$ *be a closed subscheme and* $Z' = Z\times_X X'$*. Let* $\mathcal{F}\in D_{\mathrm{ctf}}(X,\Lambda)$ *such that* $X\to S$ *is* $SS(\mathcal{F})$*-transversal and* $X\backslash Z\to Y$ *is* $SS(\mathcal{F}|_{X\backslash Z})$*-transversal. Assume that* f *and* g *are smooth morphisms and that* i_X *is properly* $SS(\mathcal{F})$*-transversal. Assume* S *(resp.* S'*) is connected of dimension* s *(resp.* s'*). Then we have*

$$i_X^!cc^Z_{X/Y/S}(\mathcal{F}) = cc^{Z'}_{X'/Y'/S'}(i_X^*\mathcal{F}) \quad \text{in} \quad CH_{s'}(Z'), \tag{3.18.2}$$

where $i_X^!\colon CH_s(Z)\to CH_{s'}(Z')$ *is the refined Gysin pull-back.*

We don't know how to remove the assumption that f is smooth and i_X is properly $SS(\mathcal{F})$-transversal.

Proof. Consider the following commutative diagram

$$\begin{array}{ccccc} T^*S'\times_{S'}X' & \xrightarrow{dg'_{X'}} & T^*Y'\times_{Y'}X' & \xrightarrow{df'} & T^*X' \\ \uparrow{\scriptstyle d\delta} & \square & \uparrow{\scriptstyle di_Y} & \square & \uparrow{\scriptstyle di_X} \\ T^*S\times_S X' & \xrightarrow{dg_{X'}} & T^*Y\times_Y X' & \xrightarrow{df_{X'}} & T^*X\times_X X' \\ \downarrow{\scriptstyle 1\times i_X} & & \downarrow{\scriptstyle 1\times i_X} & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ T^*S\times_S X & \xrightarrow[dg_X]{} & T^*Y\times_Y X & \xrightarrow[df]{} & T^*X. \end{array} \tag{3.18.3}$$

By [11, Theorem 7.6], we have

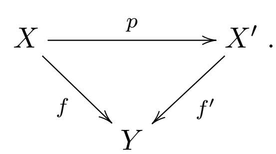
$$CC(i^*\mathcal{F}) = di_{X*}\mathrm{pr}_1^! CC(\mathcal{F}) \cdot (-1)^{-\dim(X')+\dim(X)}. \tag{3.18.4}$$

Now the result follows from the following identities:

$$\begin{aligned} cc^{Z'}_{X'/Y'/S'}(i_X^*\mathcal{F}) &= (-1)^{s'} \cdot dg'^!_{X'}(df'^! CC(i_X^*\mathcal{F})|_{T^*Y'\times_{Y'}Z'}) \\ &= (-1)^{s'-\dim(X')+\dim(X)} \cdot dg'^!_{X'}(df'^!(di_{X*}\mathrm{pr}_1^! CC(\mathcal{F}))|_{T^*Y'\times_{Y'}Z'}) \\ &= (-1)^{s} \cdot d\delta_* dg^!_{X'}(df^!_{X'}\mathrm{pr}_1^! CC(\mathcal{F})|_{T^*Y\times_Y Z'}) \\ &= (-1)^{s} \cdot d\delta_*(1\times i_X)^! dg^!_X(df^! CC(\mathcal{F})|_{T^*Y\times_Y Z}) = i^!_X cc^Z_{X/Y/S}(\mathcal{F}). \end{aligned} \tag{3.18.5}$$

□

3.19. Let $g: Y \to S$ be a smooth morphism in Sm_k. Consider a commutative diagram in Sm_k:

(3.19.1)


Let $Z \subseteq X$ be a closed subscheme. Let $\mathcal{F} \in D_{\mathrm{ctf}}(X,\Lambda)$ such that $X \to S$ is $SS(\mathcal{F})$-transversal and that $X\backslash Z \to Y$ is $SS(\mathcal{F}|_Z)$-transversal. Assume p is a proper morphism and put $Z' = p(Z)$. By [11, Lemma 3.8 and Lemma 4.2.6], the morphism $X' \to S$ is $SS(Rp_*\mathcal{F})$-transversal and that $X'\backslash Z' \to Y$ is $SS(Rp_*\mathcal{F}|_Z)$-transversal. Then we have well defined classes $cc^Z_{X/Y/S}(\mathcal{F}) \in \mathrm{CH}_s(Z)$ and $cc^{Z'}_{X'/Y/S}(Rp_*\mathcal{F}) \in \mathrm{CH}_s(Z')$.

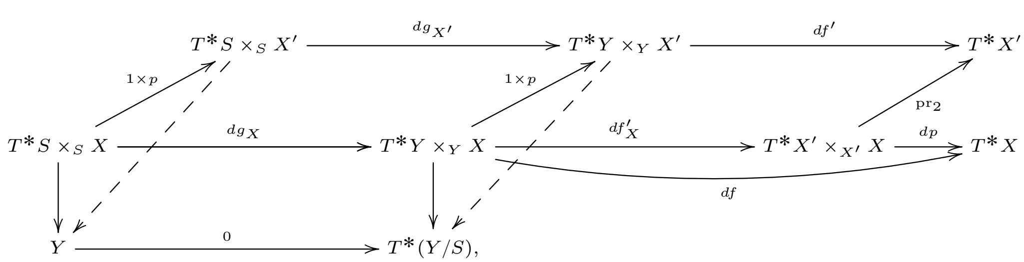
Proposition 3.20. *Consider the assumptions in 3.19. Assume moreover* $\dim p_\circ SS(\mathcal{F}) \leqslant \dim X'$*, Y is projective and p is quasi-projective. Then we have*

$$p_* cc^Z_{X/Y/S}(\mathcal{F}) = cc^{Z'}_{X'/Y/S}(Rp_*\mathcal{F}), \tag{3.20.1}$$

where $p_*: \mathrm{CH}_s(Z) \to \mathrm{CH}_s(Z')$ is the proper push-forward.

We don't know how to remove the assumptions that $\dim p_\circ SS(\mathcal{F}) \leqslant \dim X'$, Y is projective and p is quasi-projective.

Proof. Consider the following commutative diagram

(3.20.2)


where squares are cartesian diagrams. By [12, Theorem 2.2.5], we have an equality in $Z_{\dim X'}(p_\circ SS(\mathcal{F}))$:

$$CC(Rp_*\mathcal{F}) = \mathrm{pr}_{2*} dp^!(CC(\mathcal{F})). \tag{3.20.3}$$

Then we have

$$
\begin{aligned}
cc^{Z'}_{X'/Y/S}(Rp_*\mathcal{F}) &= (-1)^s \cdot dg^!_{X'}(df'^!(CC(Rp_*\mathcal{F})|_{T^*Y\times_Y Z})) \\
&\overset{(3.20.3)}{=} (-1)^s \cdot dg^!_{X'}(df'^!(\mathrm{pr}_{2*}dp^!CC(\mathcal{F})|_{T^*Y\times_Y Z})) \\
&= (-1)^s \cdot dg^!_{X'}(1\times p)_*((df'^!_X dp^!CC(\mathcal{F}))|_{T^*Y\times_Y Z}) \\
&= (-1)^s \cdot dg^!_{X'}(1\times p)_*(df^!CC(\mathcal{F})|_{T^*Y\times_Y Z}) \\
&= (-1)^s \cdot (1\times p)_* dg^!_X(df^!CC(\mathcal{F})|_{T^*Y\times_Y Z}) = p_* cc^Z_{X/Y/S}(\mathcal{F}),
\end{aligned}
\tag{3.20.4}
$$

which proves the equality (3.20.1). ☐

Corollary 3.21 (Saito, [12, Theorem 2.2.3])**.** *Let $f : X \to Y$ be a projective morphism of smooth schemes over a perfect field k, and let $y \in Y$ be a closed point. Let $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$. Assume Y is a smooth and connected curve and that f is properly $SS(\mathcal{F})$-transversal outside X_y. Then we have*

$$
-a_y(Rf_*\mathcal{F}) = f_* cc^{X_y}_{X/Y/k}(\mathcal{F}). \tag{3.21.1}
$$

Proof. By Proposition 3.20 and Theorem 3.12, we have

$$
f_* cc^{X_y}_{X/Y/k}(\mathcal{F}) \overset{(3.20.1)}{=} cc^{\{y\}}_{Y/Y/k}(Rf_*\mathcal{F}) \overset{(3.12.1)}{=} -\mathrm{dimtot} R\Phi_{\bar{y}}(Rf_*\mathcal{F}, \mathrm{id}) = -a_y(Rf_*\mathcal{F}). \tag{3.21.2}
$$

☐

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