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Real solution isolation with multiplicity of zero-dimensional triangular systems

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Abstract Existing algorithms for isolating real solutions of zero-dimensional polynomial systems do not compute the multiplicities of the solutions. In this paper, we define in a natural way the multiplicity of solutions of zero-dimensional triangular polynomial systems and prove that our definition is equivalent to the classical definition of local (intersection) multiplicity. Then we present an effective and complete algorithm for isolating real solutions with multiplicities of zero-dimensional triangular polynomial systems using our definition. The algorithm is based on interval arithmetic and square-free factorization of polynomials with real algebraic coefficients. The computational results on some examples from the literature are presented.

Keywords real solution isolation, local multiplicity, local ring, polynomial system solving

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1 Introduction

Real solution isolation for polynomials/zero-dimensional polynomial systems/semi-algebraic systems is one of the central topics in computational real algebra and computational real algebraic geometry, which has many applications in various problems with different backgrounds.

The so-called real root/zero/solution isolation of a polynomial/zero-dimensional polynomial system/semi-algebraic system with k distinct real solutions is to compute k disjoint intervals/"boxes" containing the k solutions, respectively. To our knowledge, designing algorithms for real root isolation for polynomials with rational coefficients was initiated by [1] in 1976, which was closely related to the implementation of CAD algorithm [2]. Designing and implementation of such algorithms have been deeply developed by many subsequent work [3–7] since then. Those algorithms are mainly based on Descartes' rule of sign or Vincent's theorem.

To generalize the real root isolation algorithms for polynomials to zero-dimensional triangular polynomial systems, one must consider real root isolation for polynomials with real algebraic coefficients. There are indeed some work to generalize Descartes' rule of sign to polynomials with algebraic coefficients. However, dealing with algebraic coefficients directly may affect efficiency greatly.

In [8, 9] we considered real solution isolation for semi-algebraic systems with finite solutions. We introduced a method which always enables us to avoid handling directly polynomials with algebraic

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coefficients and to deal with polynomials with rational coefficients only. A recent algorithm in [10] can compute the parity of the solutions as well as isolate real roots of zero-dimensional triangular polynomial systems.

In this paper, we define in a natural way the multiplicity of solutions of zero-dimensional triangular polynomial systems and prove that our definition is equivalent to the classical definition of local (intersection) multiplicity. Then we present an effective and complete algorithm for isolating real solutions with multiplicities of zero-dimensional triangular polynomial systems using our definition. The algorithm is based on square-free factorization of polynomials with real algebraic coefficients and our previous work [9]. We also provide computational results on some examples from the literature.

In this paper, all polynomials are in $\mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_n]$ if not specified.

2 Multiplicities of zeros of triangular sets

First, let's recall the definition of local (intersection) multiplicity. We follow the notations in Chapter 4 of [11]. Although some notations and definitions can be stated in a more general way, we restrict ourselves to the ring $\mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_n]$ since we are interested in the complex or real zeros of zero-dimensional polynomial systems.

For $p = (\eta_1, \ldots, \eta_n) \in \mathbb{C}^n$, we denote by M_p the maximal ideal generated by $\{x_1 - \eta_1, \ldots, x_n - \eta_n\}$ in $\mathbb{C}[X]$, and write

$$\mathbb{C}[X]_{M_p} = \left\{ \frac{f}{g} : f, g \in \mathbb{C}[X], g(\eta_1, \ldots, \eta_n) \neq 0 \right\}.$$

It is well-known that $\mathbb{C}[X]_{M_p}$ is the so-called local ring.

Definition 1 [11]**.** Suppose I is a zero-dimensional ideal in $\mathbb{C}[X]$ and $p \in \mathrm{Zero}(I)$, the zero set of I in $\mathbb{C}$. Then the multiplicity of p as a point in $\mathrm{Zero}(I)$ is defined to be

$$\dim \mathbb{C}[X]_{M_p} / I\mathbb{C}[X]_{M_p}.$$

That is, the multiplicity of p is the dimension of the quotient space $\mathbb{C}[X]_{M_p} / I\mathbb{C}[X]_{M_p}$ as a vector space over $\mathbb{C}$.

For a zero of a zero-dimensional triangular set, there can be a natural and intuitional definition of multiplicity as follows.

Definition 2. For a zero-dimensional triangular system,

$$\begin{cases} f_1(x_1) = 0, \\ f_2(x_1, x_2) = 0, \\ \cdots \\ f_n(x_1, \ldots, x_n) = 0, \end{cases}$$

and one of its zeros, $\xi = (\xi_1, \ldots, \xi_n)$, the multiplicity of ξ is defined to be $\prod_{i=1}^n m_i$, where m_i is the multiplicity of $x_i = \xi_i$ as a zero of the univariate polynomial $f_i(\xi_1, \ldots, \xi_{i-1}, x_i)$ for $i = 1, \ldots, n$.

Example 1. Consider the following triangular system:

$$\begin{cases} g_1 = x_1^3 + 2x_1^5 + 7x_1^7 = 0, \\ g_2 = x_2^3 + x_2^2 + x_1 x_2 = 0, \\ g_3 = x_3^2 + x_1 x_3 + x_1 x_2 = 0. \end{cases}$$

Let's compute the local multiplicity of $(0,0,0)$ by Definition 2. The multiplicity of $x_1 = 0$, a zero of g_1, is 3. Substitute $x_1 = 0$ in f_2 and the resulted g_2 is $g_2' = x_2^3 + x_2^2$. Thus, the multiplicity of $x_2 = 0$, a zero of g_2', is 2. Finally, substitute $x_1 = x_2 = 0$ in g_3, and the resulted g_3 is $g_3' = x_3^2$. Thus, the multiplicity of $x_3 = 0$, a zero of g_3', is 2. As a result, the local multiplicity of $(0,0,0)$ is $3 \times 2 \times 2 = 12$.

In the following, we will prove that Definition 2 is equivalent to Definition 1. Many notations and results are taken from [11].

Usually, a total order that is compatible with multiplication and that satisfies $1 > x_i$ for all i's, is called a local order.

Definition 3 [12]**.** (Negative lexicographical ordering) Assume $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n_{\geqslant}$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n_{\geqslant}$. We say $X^\alpha >_{nl} X^\beta$ if

$$\exists i\ (1 \leqslant i \leqslant n) \wedge (\forall j,\ 1 \leqslant j < i \Longrightarrow \alpha_j = \beta_j) \wedge (\alpha_i < \beta_i).$$

Remark 1. The negative lexicographical ordering $>_{nl}$ is obviously a local order.

For a given order, $\mathrm{lc}(f), \mathrm{lm}(f)$ and $\mathrm{lt}(f)$ denote the leading coefficient, leading monomial and leading term of f, respectively. For a set S,

$$\mathrm{lt}(S) = \{\mathrm{lt}(f) : f \in S\}.$$

Definition 4 [11]**.** Let $R = \mathbb{C}[X]_{M_p}$ and $I \subset R$ be an ideal. A set $\{g_1, \ldots, g_m\} \subset I$ is called a standard basis for I with respect to $<_{nl}$ if

$$\langle \mathrm{lt}(I) \rangle = \langle \mathrm{lt}(g_1), \ldots, \mathrm{lt}(g_m) \rangle.$$

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n_{\geqslant}$, define $|\alpha| = \sum_i \alpha_i$. For any polynomial $g = \sum_\alpha c_\alpha X^\alpha \in \mathbb{C}[X]$ with total degree d, we will write $g^h = \sum_\alpha c_\alpha t^{d-|\alpha|} X^\alpha$ for the homogenization of g with respect to t.

Definition 5 [11]**.** For the monomials in $\mathbb{C}[t, X]$, define $t^a X^\alpha >'_{nl} t^b X^\beta$ if $a + |\alpha| > b + |\beta|$ or $a + |\alpha| = b + |\beta|$ and $X^\alpha >_{nl} X^\beta$.

It is easy to verify that $>'_{nl}$ is a monomial order over $\mathbb{C}[t, X]$.

Theorem 1 [11]**.** Let I be the ideal in $\mathbb{C}[X]_{M_p}$ generated by $G = \{g_1, \ldots, g_m\}$ and $>$ be any local order. G is a standard basis for I if and only if applying Mora normal form algorithm to each S-polynomial formed from elements of the set of homogenizations $G^h = \{g_1^h, \ldots, g_m^h\}$ yields a zero remainder.

For our purpose, we state the above criterion in another form as follows.

Theorem 2. Let notations be as in Theorem 1. G is a standard basis if and only if for any nonzero S-polynomial of g_i^h and g_j^h, denoted by S_{ij}, there exist homogeneous polynomials $U, A_1, \ldots, A_m \in \mathbb{C}[t, X]$ such that

$$US_{ij} = \sum_{l=1}^{m} A_l g_l^h, \tag{1}$$

where $\mathrm{lt}(U) = t^a$ for some a,

$$a + \deg(S_{ij}) = \deg(A_l) + \deg(g_l^h)$$

for all l whenever $A_l \neq 0$, and

$$\mathrm{lt}(A_l g_l^h) \leqslant'_{nl} \mathrm{lt}(US_{ij}).$$

Remark 2. We omit the proof of Theorem 2, which is almost the same as that of Theorem 1. The criterion in Theorem 2 is independent to any algorithms. One can use Mora normal form algorithm to get such representation as (1) for each S_{ij} if G is a standard basis.

Without loss of generality, in the rest of this section we assume $p = (0, \ldots, 0)$ is a zero of the triangular set under discussion and focus on its multiplicity. Consider the following triangular set with leading terms $c_1 x_1^{m_1}, \ldots, c_n x_n^{m_n}$ respectively w.r.t. the order $>_{nl}$:

$$T = \begin{cases} f_1(x_1) = c_1 x_1^{m_1} + t_1(x_1), \\ f_2(x_1, x_2) = c_2 x_2^{m_2} + t_2(x_1, x_2), \\ \ldots, \\ f_n(x_1, \ldots, x_n) = c_n x_n^{m_n} + t_n(x_1, \ldots, x_n), \end{cases}$$

where $t_i(x_1,\dots,x_i)$ is a polynomial in $x_1,\dots,x_i$ for $i=1,\dots,n$ and c_i's are constants. Without loss of generality, we assume c_i's are all 1 in the proof of the following proposition.

Proposition 1. Let T be as above and $I=\langle T\rangle$ the ideal generated by T in the local ring $\mathbb{C}[X]_{\langle x_1,\dots,x_n\rangle}$. Then T is a standard basis for I with respect to $>_{nl}$.

Proof. According to Theorem 2, we only need to show that every nonzero S-polynomial of each pair of $T^h=\{f_1^h,\dots,f_n^h\}$ can be represented in the form of (1).

Assume that $f_i^h=t^a x_i^{m_i}+\overline{t_i}$, $f_j^h=t^b x_j^{m_j}+\overline{t_j}$ and $a<b$. The S-polynomial, S_{ij}, of f_i^h and f_j^h is

$$S_{ij}=t^{b-a}x_j^{m_j}f_i^h-x_i^{m_i}f_j^h.$$

Let $p_1=t^{b-a}x_j^{m_j}f_i^h$ and $p_2=x_i^{m_i}f_j^h$. Under the order $<'_{nl}$, the first term of p_1 is equal to the first term of p_2. If $S_{ij}\neq 0$, there exists some ℓ such that under the order $<'_{nl}$ the ℓth term of p_1 is not equal to the ℓth term of p_2 and the kth term of p_1 is equal to the kth term of p_2 for all $1\leqslant k<\ell$. Then, the kth terms of p_1 and p_2 can be represented as $x_i^{m_i}q_k$ and $t^{b-a}x_j^{m_j}q_k$ for some q_k, respectively. Thus, f_i^h and f_j^h can be respectively rewritten as

$$f_i^h=x_i^{m_i}(t^a+Q)+\overline{f_{i2}},\ f_j^h=t^{b-a}x_j^{m_j}(t^a+Q)+\overline{f_{j2}},$$

where $Q=\sum_{k=1}^{\ell-1}q_k$ and $\overline{f_{i2}}$ and $\overline{f_{j2}}$ are the remained parts of f_i^h and f_j^h, respectively, which satisfy that

$$\mathrm{lt}(t^{b-a}x_j^{m_j}\overline{f_{i2}})\neq \mathrm{lt}(x_i^{m_i}\overline{f_{j2}}).$$

It is easy to verify that $S_{ij}=t^{b-a}x_j^{m_j}\overline{f_{i2}}-x_i^{m_i}\overline{f_{j2}}$. Then

$$\mathrm{lm}(S_{ij})=\max(\mathrm{lm}(t^{b-a}x_j^{m_j}\overline{f_{i2}}),\mathrm{lm}(x_i^{m_i}\overline{f_{j2}}))$$

under the order $<'_{nl}$. Thus,

$$(t^a+Q)S_{ij}=\overline{f_{i2}}f_j^h-\overline{f_{j2}}f_i^h.$$

Let $U=t^a+Q$, $A_j=\overline{f_{i2}}$, $A_i=\overline{f_{j2}}$ and $A_s=0$ $(s\neq i$ and $s\neq j)$. Then we have

$$US_{ij}=\sum_{s=1}^{n}A_sf_s^h,$$

and all the requirements in Theorem 2 are met. Thus T is a standard basis for $\langle T\rangle$ w.r.t $<_{nl}$.

In order to prove the equivalence of Definitions 2 and 1 about the (local) multiplicity, we need the following theorem, which can be found in [11].

Theorem 3 [11]**.** Let I be an ideal in a local ring R, and assume that dim $R/\langle \mathrm{lt}(I)\rangle$ is finite for some local order $>$. Then we have

$$\dim\ R/I=\dim\ R/\langle \mathrm{lt}(I)\rangle.$$

Theorem 4. Let notations be as above and T a zero-dimensional triangular set with a zero $p=(0,\dots,0)$. If the multiplicity of p defined by Definition 2 is $m=\prod_{i=1}^n m_i$, then the local multiplicity, defined by Definition 1, of p as a point of Zero($\langle T\rangle$) is also m.

Proof. If the multiplicity is $\prod_{i=1}^n m_i$ in the sense of Definition 2, then T can be rewritten as

$$\begin{cases} f_1(x_1)=(c_1+t_{11}(x_1))x_1^{m_1},\\ f_2(x_1,x_2)=(c_2+t_{21}(x_1,x_2))x_2^{m_2}+x_1t_{22}(x_1,x_2),\\ \dots\\ f_n(X)=(c_n+t_{n1}(X))x_n^{m_n}+\sum\limits_{i=1}^{n-1}x_it_{ni+1}(X), \end{cases}$$

where $X=(x_1,\dots,x_n)$, the $t_{ij}(X)$s are polynomials in $(x_1,\dots,x_i)$ and the $t_{i1}(X)$s do not contain constants.

Under the order $>_{nl}$, the leading monomial of $f_i(x_1, \dots, x_i)$ is $c_i x_i^{m_i}$ for $i = 1, \dots, n$. According to Proposition 1, T is a standard basis of $I = \langle T \rangle$. Thus,

$$\langle \mathrm{lt}(I) \rangle = \langle x_1^{m_1}, \dots, x_n^{m_n} \rangle.$$

Let $R = \mathbb{C}[X]_{\langle x_1, \dots, x_n \rangle}$. According to Theorem 3,

$$\dim R/I = \dim R/\langle x_1^{m_1}, \dots, x_n^{m_n} \rangle = \prod_{i=1}^{n} m_i.$$

3 Algorithm for real solution isolation with multiplicity

In this section, based on the results in last section, we present an algorithm for real root isolation with multiplicity of zero-dimensional triangular polynomial equations. That is, we not only isolate the real roots, but also compute the multiplicity of each real root by Definition 2 at the same time. In this section, the input polynomial or polynomial set to our algorithms is taken from $\mathbb{Q}[X]$.

It is well-known that there exist some efficient algorithms for real root isolation of polynomials or polynomial equations or semi-algebraic systems [1, 3–6, 8–10]. To obtain the multiplicities of the real roots at the same time, our idea is simple that is to take use of square-free factorization of polynomials with rational or algebraic coefficients. When dealing with algebraic coefficients, we make use of the idea in [8, 9] which enables us to deal with rational coefficients instead.

For the univariate case, suppose $p = \prod_{i=1}^{k} p_i^i$. Isolating the real zeros of p with multiplicity contains two main steps. One is to compute the squarefree factorization of p, the other is to isolate the real zeros of the squarefree part of p. We can use many existing tools to obtain the squarefree factorization, i.e., those p_is. Then we know at once the multiplicities of those real zeros of each p_i. In principle, we may isolate the real zeros of the squarefree part of p in two ways. One way is to isolate the real zeros of $p_1 p_2 \cdots p_k$ first and then match the zeros with p_i to obtain correct multiplicities. The other way is to isolate the real zeros of each p_i separately. However, in the latter way, we may need to compute a root gap of p first. Anyway, the univariate case can be efficiently dealt with. Therefore, we do not enter the details of such algorithms and only give a description of the input and output of such function.

Calling sequence UniIsol($f(x)$)

Input: a univariate polynomial $f(x)$.

Output: a set of elements of the form $([a, b], m)$ where $[a, b]$ is an interval containing exact one real root of $f(x) = 0$ and m is the multiplicity of the root. There are not any real roots of $f(x) = 0$ outside the intervals.

Then let us consider the multivariate case. To be more precise, we state our problem as follows: That is to isolate the real solutions with multiplicities of the following zero-dimensional triangular polynomial set:

$$T = \{f_1(x_1), f_2(x_1, x_2), \dots, f_n(x_1, \dots, x_n)\}.$$

In principle, Definition 2 suggests a naive method to compute the local multiplicity as follows. First compute all the zeros of $f_1(x_1)$ and their multiplicities by UniIsol; then "substitute" the zeros for x_1 in $f_2(x_1, x_2)$ one by one, and compute all the zeros of the resulted $f_2(\bar{x_1}, x_2)$ and their multiplicities by UniIsol again, and so on. Of course, in general we cannot directly substitute the zeros in those polynomials because they may be algebraic numbers of high degrees. Nevertheless, this naive method is the main framework of our algorithm.

Let $T_i = \{f_1(x_1), f_2(x_1, x_2), \dots, f_i(x_1, \dots, x_i)\}$. We will call

$$([a_1, b_1], \dots, [a_i, b_i]) \quad \text{or} \quad ([a_1, b_1], \dots, [a_i, b_i], m)$$

an interval solution (with multiplicity m) of T_i if the "box" $[a_1, b_1] \times \cdots \times [a_i, b_i]$ contains exact one real solution of T_i (and m is the multiplicity of the solution). If T_i has k distinct real solutions, a set of k

interval solutions of T_i containing respectively the k real solutions is called a solution set of T_i. For an interval solution $r = ([a_1, b_1], \ldots, [a_i, b_i])$, we define $N_r = [x_1 - a_1, b_1 - x_1, \ldots, x_i - a_i, b_i - x_i]$ and $N_r \geqslant 0$ stands for $a_1 \leqslant x_1 \leqslant b_1, \ldots, a_i \leqslant x_i \leqslant b_i$, i.e., $(x_1, \ldots, x_i) \in [a_1, b_1] \times \cdots \times [a_i, b_i]$.

Suppose we already have a solution set of T_i and

$$(\xi_1, \ldots, \xi_i) \in [a_1, b_1] \times \cdots \times [a_i, b_i]$$

is a real root of T_i with multiplicity m. To isolate the real zeros of $f_{i+1}(\xi_1, \ldots, \xi_i, x_{i+1})$ with multiplicity, we need to

i) compute the algebraic squarefree factorization of $f_{i+1}(\xi_1, \ldots, \xi_i, x_{i+1})$, and

ii) isolate the real zeros of the squarefree part computed.

Let us first consider the second task, i.e., how to isolate the real zeros of $f_{i+1}(\xi_1, \ldots, \xi_i, x_{i+1})$ if it is squarefree. In [9], we proposed a complete algorithm, called RealZeros, for isolating the real solutions (without multiplicities) of semi-algebraic systems. Our second task can be accomplished by a sub-algorithm of RealZeros. The key idea of the sub-algorithm is to compute two suitable polynomials $\overline{f_{i+1}}$ and $\underline{f_{i+1}}$ in x_{i+1} with rational coefficients such that

$$\underline{f_{i+1}} < f_{i+1}(\xi_1, \ldots, \xi_i, x_{i+1}) < \overline{f_{i+1}}$$

by using interval arithmetic and those intervals $[a_1, b_1], \ldots, [a_i, b_i]$. One can isolate the real zeros of $f_{i+1}(\xi_1, \ldots, \xi_i, x_{i+1})$ through isolating the real zeros of $\overline{f_{i+1}}$ and $\underline{f_{i+1}}$. Therefore, we can avoid dealing with polynomials with algebraic coefficients directly. In the following, we call this sub-algorithm AlgIsol.

Calling sequence AlgIsol$(g(x_1, \ldots, x_{i+1}), T_i, r)$

Input: a squarefree polynomial $g(x_1, \ldots, x_{i+1})$, a zero-dimensional triangular polynomial set T_i as above and an interval solution $r = ([a_1, b_1], \ldots, [a_i, b_i])$ which contains exact one real zero $(\xi_1, \ldots, \xi_i)$ of T_i.

Output: a list of isolating intervals of real zeros of $g(\xi_1, \ldots, \xi_i, x_{i+1})$.

For the detail of the algorithm AlgIsol, please refer to [9].

Now, we turn to the first task, i.e., compute the algebraic squarefree factorization of $f_{i+1}(\xi_1, \ldots, \xi_i, x_{i+1})$. One may use some existing algorithms for algebraic factorization, see [13] for example, to accomplish the task. In the following, we propose a method for algebraic squarefree factorization based on algebraic gcd computation. A key manipulation in the computation is to count real solutions of semi-algebraic systems by an algorithm RealrootCount[1)] in [14].

Calling sequence RealrootCount(F, N, P, H)

Input: a zero-dimensional polynomial set F, a list of non-strict inequalities N, a list of strict inequalities P and a list of inequations H.

Output: the number of distinct real roots of the system $\{F = 0, N \geqslant 0, P > 0, H \neq 0\}$.

Calling sequence AlgGCD(p_1, p_2, T_i, r)

Input: two polynomials p_1, p_2 in $x_1, \ldots, x_{i+1}$, a zero-dimensional triangular polynomial set T_i as above and an interval solution $r = ([a_1, b_1], \ldots, [a_i, b_i])$ of T_i.

Output: $\gcd(p_1(\xi_1, \ldots, \xi_i, x_{i+1}), p_2(\xi_1, \ldots, \xi_i, x_{i+1}))$, the greatest common divisor of p_1 and p_2 viewed as polynomials in x_{i+1} w.r.t. the interval solution r. Here $(\xi_1, \ldots, \xi_i)$ is the only real solution in r.

Step 0. Suppose the subresultant chain of p_1 and p_2 w.r.t. x_{i+1} is $S_\mu, S_{\mu-1}, \ldots, S_0$ with principal subresultant coefficients $R_\mu, R_{\mu-1}, \ldots, R_0$, respectively. Set $j \leftarrow 0$.

Step 1. Compute R_j.

1) The algorithm is called nearsolve in [14].

Step 2. If RealrootCount$(T_i, N_r, [\ ], [R_j]) = 0$, i.e., the interval solution makes R_j vanish, then set $j \leftarrow j+1$ and go to Step 1.

Step 3. Return S_j.

There are several mature algorithms [15, 16] for squarefree factorization of polynomials in $\mathbb{K}[x]$ where $\mathbb{K}$ is $\mathbb{Z}, \mathbb{Q}$ or a finite field. It is well known that such algorithms for univariate case only contain two main manipulation: gcd computation and polynomial division in $\mathbb{K}[x]$. If we replace the gcd computation in those algorithms with our AlgGCD computation and replace the division manipulation with pseudo-division, then those algorithms will compute algebraic squarefree factorization as we want. Therefore, we only give a simple description of our algorithm here.

Calling sequence AlgSF$(p(x_1, \ldots, x_{i+1}), T_i, r)$

Input: a polynomial p in $x_1, \ldots, x_{i+1}$, a zero-dimensional triangular polynomial set T_i as above and an interval solution $r = ([a_1, b_1], \ldots, [a_i, b_i])$ of T_i.

Output: the squarefree factorization of p viewed as a polynomial in x_{i+1} w.r.t. the interval solution r, i.e., the squarefree factorization of $p(\xi_1, \ldots, \xi_i, x_{i+1})$ where $(\xi_1, \ldots, \xi_i)$ is the only real solution in r.

Now, we are ready to describe our algorithm MultiIsolate for real solution isolation with multiplicity of zero-dimensional triangular polynomial sets.

Calling sequence MultiIsolate(T)

Input: a zero-dimensional triangular polynomial set $T = \{f_1(x_1), \ldots, f_n(x_1, \ldots, x_n)\}$.

Output: a solution set of T with multiplicity.

Step 1. $i \leftarrow 1$, $L_i \leftarrow \mathtt{UniIsol}(f_1)$.

Step 2. L_i is a solution set of T_i with multiplicity. If $i = n$, return L_n.

Step 3. For each interval solution $r = ([a_1, b_1], \ldots, [a_i, b_i])$ in L_i with multiplicity, compute AlgSF$(f_{i+1}(x_1, \ldots, x_{i+1}), T_i, r)$. Therefore, we know at once the multiplicity of each factor. Assume $\widetilde{f_{i+1}}$ is the squarefree part of f_{i+1}. Then, by applying AlgIsol$(\widetilde{f_{i+1}}, T_i, r)$ we can obtain the isolating intervals of real zeros of f_{i+1}. Therefore, it is easy to obtain a solution set L_{i+1} of T_{i+1} with multiplicity by Definition 2. $i \leftarrow i+1$ and go to Step 2.

Remark 3. Let $r = ([a_1, b_1], \ldots, [a_n, b_n])$ be an interval solution of T and $\xi = (\xi_1, \ldots, \xi_n)$ is the real solution in r. If $\mathrm{lc}(f_i)(\xi_1, \ldots, \xi_{i-1}) \neq 0$ for $2 \leqslant i \leqslant n$, T is said to be regular w.r.t. ξ (or r). If $f_i(\xi_1, \ldots, \xi_{i-1}, x_i)$ is squarefree for $1 \leqslant i \leqslant n$, T is said to be squarefree w.r.t. ξ (or r).

It is clear that MultiIsolate(T) actually computes as well a regular and squarefree decomposition of the given triangular set T w.r.t. its real zeros, respectively. That is to say, we compute a set of triangular sets W_j and their solution sets Q_j such that $\cup_j Q_j$ is a solution set of T and each W_j is regular and squarefree w.r.t. each solution in Q_j. If we modify slightly the algorithm MultiIsolate, we can output the regular and squarefree decomposition.

Now, let's illustrate the main steps of the algorithm MultiIsolate by a simple example.

Example 2. Consider the following triangular equations

$$\begin{cases} f_1 = (x^2-3)^4, \\ f_2 = y^3 - (x^2-3)y. \end{cases}$$

Step 1. $i = 1, n = 2$. UniIsol(f_1) returns a solution set of f_1 as follows:

$$\{([13/8, 7/4], 4),\ ([-7/4, -13/8], 4)\}.$$

That means f_1 has two distinct real solutions contained respectively in the two intervals. Both of the two solutions are of multiplicity 4.

Please note that the solutions are represented by intervals with rational endpoints. The width of the interval can be decreased arbitrarily on demand. By default, the algorithm will output the results when the intervals do not intersect. Therefore, there is no approximate computation here and in principle the algorithm does not need a precision input.

Step 2. Setting L_1 to be the solution set, because $i < n$, we continue and go to step 3.

Step 3. For the first solution $[13/8, 7/4]$, by calling

$$\mathrm{AlgSF}(f_2,\ [f_1],\ [13/8, 7/4]),$$

we get a square-free factorization of f_2 w.r.t. the first solution $[13/8, 7/4]$ to the triangular set $[f_1]$, which is y^3. Therefore we know at once the multiplicity of the solutions to the square-free part, i.e., y, is 3.

Then, by calling

$$\mathrm{AlgIsol}(y, [f_1], [13/8, 7/4]),$$

we obtain a solution $[0, 0]$ to the second equation. Therefore, by Definition 2, we know that $[[13/8, 7/4], [0, 0]]$ is a solution "box" to the system of multiplicity $3 \times 4 = 12$.

Similarly, for the second solution in L_1, we obtain another solution "box", $[[-7/4, -13/8], [0, 0]]$, of multiplicity 12.

Now, $i \leftarrow i + 1$. Because $i = n$, we are done.

At the same time, we also obtain a regular and square-free decomposition of the input system as $[x^2 - 3, y]$.

4 Examples

The algorithm MultiIsolate has been implemented as a Maple program which is included in our package DISCOVERER [17]. For an input zero-dimensional triangular system, our program can compute the real solution isolation of the system with multiplicity and output a regular and squarefree decomposition (see Remark 3) of the system w.r.t. those real solutions. Our program can detect whether the input system is zero-dimensional. If it is not, the program will return a message: "The dimension of the system is positive."

In this section, we illustrate the function of our program by some examples. The timings are collected on a Thinkpad X200 running Maple 11 with 2.4GHz CPU, 1G memory and Windows Vista by using the time command in Maple.

Example 3. Consider the following triangular system:

$$\begin{cases} f_1 = x - 2, \\ f_2 = (x + y - 3)^3 (y + 3), \\ f_3 = (yz^2 + xz + 1)^2 ((x - y)^4 z + x - y). \end{cases}$$

Within 1.6 s, our program outputs a solution set as follows:

$$\left[\left[\left[[2,2],[-3,-3],\left[-\frac{1}{4},0\right]\right],1\right],\quad [[[2,2],[-3,-3],[1,1]],2],\right.$$
$$\left.\left[\left[[2,2],[-3,-3],\left[-\frac{1}{2},-\frac{1}{4}\right]\right],2\right],\quad [\ [[2,2],[1,1],[-1,-1]],15\ ]\ \right].$$

That means the system has four real solutions which are of multiplicities $1, 2, 2, 15$, respectively. Our program also outputs a regular and squarefree decomposition of the system w.r.t. the four distinct real solutions respectively as follows:

$$[x - 2, y + 3, 1 + 125z], \quad [x - 2, y + 3, -1 + 3z^2 - 2z], \quad [x - 2, y - 1, z + 1].$$

Note that the second and third solutions are both solutions to the second equations above.

Example 4. Consider the following triangular system:

$$\begin{cases} f_1 = (x+1)(x-2), \\ f_2 = (x-y+1)^2(y-5)+(y-3)x, \\ f_3 = (xy-6)z^2+2z+1. \end{cases}$$

The system has seven real solutions all of multiplicities 1. The computation cost is 0.7 s.

$$\begin{aligned} &[[[[2,2],[3,3],[-1/2,-1/2]],1], \quad [[[-1,-1],[-1,-3/4],[-3/8,-1/8]],1], \\ &[[[-1,-1],[-1,-3/4],[3/8,7/8]],1], \quad [[[-1,-1],[1/2,3/4],[-3/8,-1/8]],1], \\ &[[[-1,-1],[1/2,3/4],[3/8,3/4]],1], \quad [[[-1,-1],[5,21/4],[-3/8,-1/8]],1], \\ &[[[-1,-1],[5,21/4],[1/4,1/2]],1]]. \end{aligned}$$

A regular and squarefree decomposition is

$$[x-2,f,1+2z], \ [x+1,f,g],$$

where $f = x^2y-5x^2-2xy^2+13xy-13x+y^3-7y^2+11y-5$, $g = yz^2x-6z^2+2z+1$.

Example 5. The following triangular system is taken from [18]:

$$\begin{cases} f_1 = x^4, \\ f_2 = x^2y+y^4, \\ f_3 = z+z^2-7x^3-8x^2. \end{cases}$$

Within 0.1 s, we obtain two distinct real roots with multiplicities 16.

$$[[[0,0],[0,0],[-1,-1]],16], \quad [[[0,0],[0,0],[0,0]],16].$$

And a regular and squarefree decomposition is

$$[x,y,z+z^2-7x^3-8x^2].$$

Example 6. The following triangular system is taken from [10]:

$$\begin{cases} f_1 = x^4-3x^2-x^3+2x+2, \\ f_2 = y^4+xy^3+3y^2-6x^2y^2+4xy+2xy^2 \\ \qquad -4x^2y+4x+2. \end{cases}$$

The time for computation is 3.6 s and we obtain 12 distinct real roots.

$$\begin{aligned} &[[[51/32,13/8],[-119/32,-475/128]],1], \quad [[[51/32,13/8],[-147/128,-145/128]],1], \\ &[[[51/32,13/8],[53/64,107/128]],1], \quad [[[51/32,13/8],[307/128,77/32]],1], \\ &[[[-5/8,-19/32],[-3/8,1/4]],1], \quad [[[-5/8,-19/32],[13/8,17/8]],1], \\ &[[[45/32,23/16],[-3025/1024,-1499/512]],1], \quad [[[45/32,23/16],[-1347/1024,-2639/2048]],1], \\ &[[[45/32,23/16],[11/8,3/2]],2], \quad [[[-23/16,-45/32],[-5/8,-1/8]],1], \\ &[[[-23/16,-45/32],[17/4,5]],1], \quad [[[-23/16,-45/32],[-3/2,-11/8]],2]. \end{aligned}$$

It is clear that two of the solutions are of multiplicities 2 and the others are of multiplicities 1. With respect to those solutions, we have a regular and squarefree decomposition as follows:

$$[x^2-x-1,h_1], \ [x^2-2,h_2], \ [x^2-2,h_3],$$

where

$$
\begin{aligned}
h_1 &= y^4 + xy^3 + 3y^2 - 6x^2y^2 + 4xy + 2xy^2 - 4x^2y + 4x + 2,\\
h_2 &= -23354573041809 - 9122537689096xy^2 + 39406733143725xy + 17148617740054x +\\
&\quad 13135577714575y^2 - 54735226134576y,\\
h_3 &= -104xy + 335y - 335x + 208.
\end{aligned}
$$

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