Real Solution Isolation Based on Interval Krawczyk Operator

Ting Zhang, Rong Xiao, Bican Xia
LMAM & School of Mathematical Sciences, Peking University
Beijing 100871, China
ztfriend@acca.com.cn, rxiao@pku.edu.cn, xbc@math.pku.edu.cn

1 Statement of the problem

Let $\mathbb{Q}$ denote rational numbers, $\mathbb{R}$ denote real numbers and $\mathbb{Q}[X]$ denote the ring of polynomials in $(x_1, \ldots, x_n)$ with coefficients from $\mathbb{Q}$. $[a, b] = \{x \mid a \leq x \leq b, x \in \mathbb{R}\}$ is called an *interval* and an *interval vector* is a vector with interval elements. An interval vector with n elements is indeed an n-dimensional cube in $\mathbb{R}^n$.

A set of n polynomials in $\mathbb{Q}[X]$, $\{f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)\}$, is called a *normal zero dimensional algebraic system* (NZAS for short), if the following algebraic system

$$\begin{cases} f_1(x_1, x_2, \ldots, x_n) = 0, \\ f_2(x_1, x_2, \ldots, x_n) = 0, \\ \qquad\quad \ddots \\ f_n(x_1, x_2, \ldots, x_n) = 0, \end{cases} \tag{1}$$

has only finite many real solutions.

Our problem is to isolate the real solutions (in a given cube) of normal zero dimensional systems. That is to say, for a given NZAS in the form of (1) having k distinct real solutions, compute k interval vectors (cubes) such that each cube contains only one solution and no two cubes intersect.

2 Importance of the problem

Solving algebraic equations is no doubt one of the most important problems in mathematics and other related fields. In the viewpoint of symbolic computation, the solutions to algebraic equations should be "exact" and not be "approximate". That is to say, solving the equations numerically is not our goal and we need to isolate the real solutions. In fact, real solution isolation of one equation or several equations forms a base of many algorithms in the field of computational real algebraic geometry.

3 Contribution to the problem

Based on a modified Krawczyk-Moore's interval operator, we propose a general and practical symbolic algorithm named NRoots, which can isolate all the real solutions to an NZAS in a given interval vector. The correctness of the algorithm is proven.

In order to isolate *all* real solutions of a given NZAS, we extend the algorithm NRoots by combining interval value estimation with the method of resultant.

4 Originality of the contribution

There are many works related to real solution isolation of one polynomial or a polynomial system based on different principles, see, for example, [2, 3, 4, 6, 7, 8, 10].

Existing symbolic algorithms for isolating real solutions of polynomial equations require triangularizing the equations first, which may involve very heavy computation in some cases. Our method handles the input systems directly by interval iteration, so a better performance can be expected on those systems that are hard to be triangularized, such as sparse systems with high degrees or randomly generated systems.

The original Krawczyk-Moore interval iteration method [1, 5, 9] for solving nonlinear system is a numerical one, which has no guarantee that the outputs are exact solution intervals and no solutions missed. Although it performs symbolic and numerical computation, our method is essentially a symbolic algorithm whose correctness is proven. And we can accelerate the exclusion of "bad intervals" (which contain no roots) by combination of interval evaluation and original testing.

We find that better interval evaluation and tight root bounds for multivariate polynomials will improve the efficiency greatly. We also adopt some technics to improve the performance of our algorithms.

5 Non-triviality of the contribution

Our method has been implemented as a Maple program, which solves many examples [11]. For instance, consider the following polynomial system generated randomly by Maple:

$$\begin{cases} f_1 = -14y^6z - 9y^5x^5z^3 - 51y^{13}x^4z^2 + 1 = 0, \\ f_2 = yxz^{13} + 5yx^9z^7 - 86y^6x^6z^8 + 1 = 0, \\ f_3 = 50y^8z^4 + 67yx^{16} - 39y^5x^2z^{11} + 1 = 0, \\ -1 \le x \le 1, -1 \le y \le 1, -1 \le z \le 1. \end{cases}$$

Our program runs 78.3 seconds in Maple 8 on a PC (Pentium 866Hz CUP, 256Mb memory, Win2000) and outputs the following result:

$$[[\frac{-1797}{2048}, \frac{-897}{1024}], [\frac{-61}{64}, \frac{-973}{1024}], [\frac{677}{1024}, \frac{341}{512}]],$$

$$[[\frac{28311}{32768},\frac{14159}{16384}],[\frac{-1829}{2048},\frac{-1827}{2048}],[\frac{45057}{65536},\frac{2819}{4096}]],$$
$$[[\frac{3449}{4096},\frac{1727}{2048}],[\frac{-747}{1024},\frac{-1487}{2048}],[\frac{821}{1024},\frac{825}{1024}]].$$

References

[1] G., Alefeld and J., Herzberger: *Introduction to interval computations* (translated by Jon Rokne). Academic Press, New York, 1983.

[2] G. E., Collins, J. R., Johnson and W., Krandick: Interval Arithmetic in Cylindrical Algebraic Decomposition. *J. Symb. Comput.*, Vol 34, Issue 2, pp.145–157, 2002.

[3] G. E., Collins and R., Loos: Real Zeros of Polynomials. In: *Computer Algebra Symbolic and Algebraic Computation* (B., Buchberger, G. E., Collins, R., Loos, eds.), pp.83–94, Springer-Verlag, 1983.

[4] Dipl.-Inf. Volker Stahl: Interval Methods for Bounding the Range of Polynomials and Solving Systems of Nonlinear Equations. PhD thesis, University of Linz, Austria, 1995.

[5] E. R., Hansen: A globally convergent interval method for computing and bounding real roots. *BIT 18*, pp.415–424, 1978.

[6] J. R., Johnson: Real Algebraic Number Computation Using Interval Arithmetic. In: *Proc. ISSAC'92* (P. S., Wang ed.), pp.195–205, ACM Press, 1992.

[7] J. R., Johnson: Algorithms for Polynomial Real Root Isolation. In: *Quantifier Elimination and Cylindrical Algebraic Decomposition* (B. F., Caviness and J. R. Johnson eds.), pp.269–299, Springer-Verlag, 1998.

[8] F. Rouillier and P., Zimmermann: Efficient isolation of a polynomial real roots. INRIA, Research Report No.4113, 2001.

[9] D. R., Wang, L. S., Zhang and N. Y., Deng: *Interval methods for nonlinear equations solving*(in Chinese). Shanghai scientific and technical publishers, 1986.

[10] B. C., Xia and L., Yang: An algorithm for isolating the real solutions of semi-algebraic systems. *J. Symb. Comput.*, Vol 34, pp.461–477, 2002.

[11] T., Zhang: Real solution isolation of a class of algebraic systems. Dissertation, Peking University, 2004.