Solving Parametric Semi-Algebraic Systems

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1 Statement of the problem

Let $\mathcal{Q}$ be the field of rational numbers and $\mathcal{Q}[u_1, ..., u_d, x_1, ..., x_s]$ the ring of polynomials in n indeterminates with coefficients in $\mathcal{Q}$ and $d + s = n$ ($0 \leq d < n$). The indeterminates are divided into two groups: $\mathbf{u} = (u_1, ..., u_d)$ and $\mathbf{x} = (x_1, ..., x_s)$, which are called parameters and variables, respectively. A *polynomial set* is a finite set of nonzero polynomials in $\mathcal{Q}[\mathbf{u}, \mathbf{x}]$. The following system is called a *parametric semi-algebraic system* (SAS for short).

$$\begin{cases} p_1(\mathbf{u}, \mathbf{x}) = 0, ..., p_r(\mathbf{u}, \mathbf{x}) = 0, \\ g_1(\mathbf{u}, \mathbf{x}) \geq 0, ..., g_k(\mathbf{u}, \mathbf{x}) \geq 0, \\ g_{k+1}(\mathbf{u}, \mathbf{x}) > 0, ..., g_t(\mathbf{u}, \mathbf{x}) > 0, \\ h_1(\mathbf{u}, \mathbf{x}) \neq 0, ..., h_m(\mathbf{u}, \mathbf{x}) \neq 0, \end{cases} \tag{1}$$

where $r > 1, t \geq k \geq 0, m \geq 0$ and each p_i, g_i or h_i is a polynomial in $\mathcal{Q}[\mathbf{u}, \mathbf{x}]$. An SAS in the form of (1) is usually denoted by a 4-tuple $[\mathbb{P}, \mathbb{G}_1, \mathbb{G}_2, \mathbb{H}]$ of polynomial sets where $\mathbb{P} = \{p_1, ..., p_r\}, \mathbb{G}_1 = \{g_1, ..., g_k\}, \mathbb{G}_2 = \{g_{k+1}, ..., g_t\}$ and $\mathbb{H} = \{h_1, ..., h_m\}$.

For an SAS in the form of (1), our problem is to determine the *real solution classification* of the system, *i.e.*, to determine for what parametric values the system has a prescribed number of distinct real solutions or to indicate for what parametric values the system may have positive dimensional real solutions.

2 Importance of the problem

Real solution classification of parametric semi-algebraic systems has very close relation to the problem of real quantifier elimination and has numerous applications in different fields. In fact, many problems in both theory and application can be transformed into solving semi-algebraic systems. For example, automated deduction of inequality-type theorem, stability analysis of biological systems, the Solotareff approximation problem, some problems in program verification, the "PnP" problem in computer vision, etc..

3 Contribution to the problem

For a given SAS S in the form of (1), let the variables be ordered as $x_1 \prec \cdots \prec x_s$ and the polynomials in S be considered as polynomials in $\mathcal{Q}(\mathbf{u})[\mathbf{x}]$. By any of the algorithms described in [6, 8, 17], we can transform the polynomial set $\mathbb{P} = \{p_1, ..., p_r\}$ into *regular sets* $\mathcal{T} = \{\mathbb{T}_1, ..., \mathbb{T}_e\}$. Furthermore, we can make $\mathbb{T}_i$ and $\mathbb{T}_j$ have no common zero for all $i \neq j$ and *simplicial* with respect to all constraints (inequalities and inequations) according to the method of [16, 17].

There are three possibilities for $\mathcal{T}$. (A) Each $\mathbb{T}_i$ $(1 \leq i \leq e)$ has exactly s polynomials, say, $f_{i1}, ..., f_{is}$, and each f_{ij} $(1 \leq j \leq s)$ has x_j as its leading variable. (B) Some $\mathbb{T}_i$s have less than s polynomials. (C) $\mathcal{T} = \emptyset$.

Case A. This case is a basic case and we have proposed a complete and practical algorithm for solving SASs under this case in previous papers [14, 15, 16]. In this short note, we shall focus on how our method can be generalized to handle the other two cases. With this generalization, our method becomes a complete method for solving any SASs.

Case B. Suppose $\mathbb{T}$ is a regular set with less than s polynomials. We treat some variables, say, $x_{i_1}, ..., x_{i_k}$, as parameters and apply the above-mentioned algorithm to $\mathbb{T}$. Suppose ϕ is the condition obtained. If ϕ is not identically true or false, we perform real quantifier elimination to $(\exists x_{i_1}) \cdots (\exists x_{i_k})\phi$. Here, we can use the algorithm of CAD. The resulting formula indicates the condition for the original system to have (positive dimensional) real solutions. If ϕ is identically true, the system has k-dimensional real solutions for general parametric values. If ϕ is identically false, the dimension of real solutions is less than k for general parametric values. Then, we put a "boundary" (a factor of border polynomial) into $\mathbb{T}$ and solve this new system recursively by similar procedure.

Case C. $\mathcal{T} = \emptyset$ indicates the system has no real solutions for generic $\mathbf{u}$. We treat some parameters as variables and apply the above-mentioned algorithm for Case A to S. In resulting regular sets, the polynomial equations having only parameters should be treated as "boundaries" which indicate that the system may have real solutions only when parameters are on these boundaries.

4 Originality of the contribution

There are some famous methods for solving parametric polynomial systems, such as those proposed in [3, 12, 2, 7, 10, 17, 9]. Our method takes use of the tool in [17]. The concepts of *border polynomial* and *discrimination polynomial* for SAS were introduced in our previous papers [14, 13, 15, 16] where we focused on solving SAS under Case A. In this short note, we sketch a generalization of our previous method to those SASs under Cases B and C.

5 Non-triviality of the contribution

Our algorithm has been implemented as a Maple program and applied to many of the problems mentioned in Section 2. For most of those problems with reasonable size, our program works efficiently. The heavy computations involved in our method are the triangulization and regularization of the system and a partial CAD on border polynomials. So, by "reasonable size" we mean that the system is not too difficult to be triangulization and the number of parameters is no more than 4 in general. Here, we report the performance of our algorithm on the Solotareff approximation problem [4], the Whiteney Umbrella problem [11] and Hong's problem [5] which can be transformed into SASs under Case C. Due to limitation on pages, we only report our result and timings and refer the readers to [4, 11, 5] for details of the problems.

For the Solotareff approximation problem, our program runs 14 seconds in Maple 9 on a notebook computer (Pentium 1.13 Ghz CPU with 256 M memory) and outputs the following result equivalent to that in [4]:

$$[\ r > 2 \wedge (2)R\ ] \text{ or } [\ r > 0 \wedge r < 2 \wedge r^2 - 24r + 16 < 0 \wedge (2)R\ ],$$

where $R = 324a^4 + (324r^2 - 2016)a^3 + (108r^4 - 1128r^2 + 4576)a^2 + (12r^6 - 224r^4 + 1392r^2 - 4480)a - 15r^6 + 112r^4 - 608r^2 + 1600$ and $(2)R$ means the second smallest root of R in a when r is specified.

For the Whiteney Umbrella problem, our program runs less than 0.1 seconds in Maple 9 on the same machine and outputs the following result equivalent to that in [11]:

$$[\ x^2 - y^2 z = 0 \wedge y \neq 0\ ] \text{ or } [\ x = y = 0 \wedge z \geq 0\ ].$$

For Hong's problem, our program runs 22.6 seconds and the resulting formula is of the form $[\ \phi'_1 \wedge (k_i)H\ ] \vee \cdots \vee [\ \phi'_{32} \wedge (k_i)H\ ]$, where each ϕ'_i is a conjunction of the signs of at most 9 polynomials in u and v, $H = -u^2w^2 + (2uv + u^3v - v^3)w - 3u^2v^2 + v^4u - v^2 - u^5$ and $k_i = 1$ or 2, indicating that the smallest or second smallest root of H in w when u and v are specified. This somewhat confirms the claim in [1] that "a short solution formula may not exist".

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