Quantifier Elimination for Quartics

Lu Yang[1] and Bican Xia[2,⋆]

[1] Guangzhou University, Guangzhou 510006, China
[2] School of Mathematical Sciences, Peking University, China
xbc@math.pku.edu.cn

Abstract. Concerning quartics, two particular quantifier elimination (QE) problems of historical interests and practical values are studied. We solve the problems by the theory of *complete discrimination systems* and *negative root discriminant sequences* for polynomials that provide a method for real (positive/negative) and complex root classification for polynomials. The equivalent quantifier-free formulas are obtained mainly be hand and are simpler than those obtained automatically by previous methods or QE tools. Also, applications of the results to program verification and determination of positivity of symmetric polynomials are showed.

1 Introduction

The elementary theory of real closed fields is the first-order theory with atomic formulas of the forms $A = B$ and $A > B$ where A and B are multivariate polynomials with integer coefficients and an axiom system consisting of the real closed fields axioms. The problem of quantifier elimination (QE) for real closed fields can be expressed as: for a given standard prenex formula ϕ find a standard quantifier-free formula ψ such that ψ is equivalent to ϕ. The problem of quantifier elimination for real closed field is an important problem originating from mathematical logic with applications to many significant and difficult mathematical problems with various backgrounds.

Many researchers contribute to QE problem. A. Tarski gave a first quantifier elimination method for real closed fields in 1930s though his result was published almost 20 years later [Ta51]. G. E. Collins introduced a so-called *cylindrical algebraic decomposition* (CAD) algorithm in the early 1970s [Co75] for QE problem. Since then, the algorithm and its improved variations have become one of the major tools for performing quantifier elimination. Through these years, some new algorithms have been proposed and several important improvements on CAD have been made to the original method. See, for example, [ACM84b, ACM88, Br01a, Br01b, BM05, Co98, CH91, DSW98, Hong90, Hong92]
and [Hong96, Mc88, Mc98, Re92, Wei94, Wei97, Wei98]. Most of the works including Tarski's algorithm were collected in a book [CJ98].

⋆ Corresponding author.

J. Calmet, T. Ida, and D. Wang (Eds.): AISC 2006, LNAI 4120, pp. 131–145, 2006.

In this paper, we consider the following two QE problems:

$$(\forall\lambda > 0)\ (\ \lambda^4 + p\,\lambda^3 + q\,\lambda^2 + r\lambda + s > 0\ ) \tag{1}$$

and

$$(\forall\lambda \geq 0)\ (\ \lambda^4 + p\,\lambda^3 + q\,\lambda^2 + r\lambda + s \geq 0\ ), \tag{2}$$

where $s \neq 0$.

Many researchers studied the following problem of quantifier elimination (see, for example, [AM88, CH91, La88, Wu92, Wei94]),

$$(\forall x)(x^4 + px^2 + qx + r \geq 0).$$

Problems (1) and (2) are similar to this famous QE problem but have obviously different points, that is, the variable λ has to be positive or non-negative in our problems. The two problems attract us not only because they are related to the above famous QE problem but also because we encounter them when studying some problems concerning program termination [YZXZ05] and positivity of symmetric polynomials of degree 4.

Let $Q(\lambda) = \lambda^4 + p\,\lambda^3 + q\,\lambda^2 + r\lambda + s$ with $s \neq 0$[1]. Problem (1) is equivalent to finding the necessary and sufficient condition such that $Q(\lambda)$ does not have positive zeros and Problem (2) is equivalent to finding the necessary and sufficient condition such that $Q(\lambda)$ does not have non-negative zeros or the non-negative zeros of $Q(\lambda)$ (if any) are all of even multiplicities. Therefore, if one has an effective tool for root classification or positive-root-classification of polynomials, the problems can be solved in this way which is different from existing algorithms for QE.

There do exist such tools. Actually, one can deduce such a method from the Chapters 10 and 15 of Gantmacher's book [Ga59] in 1959[2]. González-Vega etc. proposed a theory on root classification of polynomials in [GLRR89] which is based on the Sturm-Habicht sequence and the theory of subresultants. For QE problems in the form $(\forall x)(f(x) > 0)$ or $(\forall x)(f(x) \geq 0)$ where the degree of $f(x)$ is a positive even integer, González-Vega proposed a combinatorial algorithm [Gon98] based on the work in [GLRR89]. Other applications of the theory in [GLRR89] to QE problems in the form $(\forall x > 0)(f(x) > 0)$ and other variants in the context of control system design were studied by Anai etc., see [AH00] for example.

The authors also have such kind of tools [YHZ96, Yang99, YX00] at hand. The theory of *complete discrimination systems* for polynomials proposed in [YHZ96] and the *negative root discriminant sequences* for polynomials proposed in [Yang99, YX00] are just appropriate tools for root classification and positive-root-classification of polynomials[3]. With the aid of these tools, determining the

[1] If $s = 0$, the problems essentially degenerate to similar problems with polynomials of degree 3 which are much easier.

[2] The Russian version of the book is published in 1953.

[3] The theories in [GLRR89] and [YHZ96] are both essentially based on the relations between subresultant chains and Sturm sequences (or Sturm-Habicht sequences), *i.e.*,

number of real (or complex/positive) zeros of a polynomial f is reduced to discussing the number of sign changes in a list of polynomials in the coefficients of f (see also Section 2 of this paper for details). There are many research works making heavy use of complete discrimination systems, see for example [WH99, WH00, WY00].

Solutions to those two problems presented in this paper are obtained mainly by hand with some computation by computer. Our formulas, especially for the semi-definite case (Problem 2), are simpler than those generated automatically by previous methods or QE tools and thus make them possible for AI applications. Hopefully, our "manual" method presented here could be turned into a systematic algorithm later on.

The rest of the paper is organized as follows. Section 2 devotes to some basic concepts and results concerning complete discrimination systems and negative root discriminant sequences for polynomials. Section 3 presents our solutions to Problems (1) and (2). Applications of our results to program termination and determination of positivity of symmetric polynomials are showed in Section 4.

2 Preliminaries

For convenience of readers, in this section we provide preliminary definitions and theorems (without proof) concerning complete discrimination systems and negative root discriminant sequences for polynomials. For details, please be referred to [YHZ96, Yang99, YX00].

Definition 1. *Given a polynomial with general symbolic coefficients* $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$, *the following* $(2n+1) \times (2n+1)$ *matrix is called the* discrimination matrix *of* $f(x)$ *and denoted by* $\mathrm{Discr}(f)$.

$$\begin{bmatrix}
a_0 & a_1 & a_2 & \cdots & a_n & & & & & \\
0 & na_0 & (n-1)a_1 & \cdots & a_{n-1} & & & & & \\
 & a_0 & a_1 & \cdots & a_{n-1} & a_n & & & & \\
 & 0 & na_0 & \cdots & 2a_{n-2} & a_{n-1} & & & & \\
 & & & \cdots & \cdots & & & & & \\
 & & & \cdots & \cdots & & & & & \\
 & & & & a_0 & a_1 & \cdots & \cdots & a_n & \\
 & & & & 0 & na_0 & \cdots & \cdots & a_{n-1} & \\
 & & & & & a_0 & a_1 & \cdots & \cdots & a_n
\end{bmatrix}$$

Denote by d_k $(k = 1, 2, \cdots, 2n+1)$ the determinant of the submatrix of $\mathrm{Discr}(f)$ formed by the first k rows and the first k columns.

Definition 2. *Let* $D_k = d_{2k}, k = 1, \cdots, n$. *We call* $[D_1, \cdots, D_n]$ *the* discriminant sequence *of* $f(x)$ *and denote it by* $\mathrm{DiscrList}(f, x)$. *Furthermore, we call* $[d_1d_2, d_2d_3, \cdots, d_{2n}d_{2n+1}]$ *the* negative root discriminant sequence *of* $f(x)$ *and denote it by* n.r.d.(f).

based on the *subresultant theorem.* However, the main results in these two theories are expressed in different forms.

Definition 3. *We call* $[\text{sign}(B_1), \text{sign}(B_2), \cdots, \text{sign}(B_n)]$ *the* sign list *of a given sequence* $[B_1, B_2, \cdots, B_n]$.

Definition 4. *Given a sign list* $[s_1, s_2, \cdots, s_n]$*, we construct its* revised sign list $[t_1, t_2, \cdots, t_n]$ *as follows:*

– *If* $[s_i, s_{i+1}, \cdots, s_{i+j}]$ *is a section of the given list, where*

$$s_i \neq 0, s_{i+1} = \cdots = s_{i+j-1} = 0, s_{i+j} \neq 0,$$

then, we replace the subsection $[s_{i+1}, \cdots, s_{i+j-1}]$ *by the first* $j-1$ *terms of* $[-s_i, -s_i, s_i, s_i, -s_i, -s_i, s_i, s_i, \cdots]$*, i.e., let*

$$t_{i+r} = (-1)^{[(r+1)/2]} \cdot s_i, \quad r = 1, 2, \cdots, j-1.$$

– *Otherwise, let* $t_k = s_k$*, i.e., no changes for other terms.*

For example, the revised one of the sign list $[1, 0, 0, 0, 1, -1, 0, 0, 1, 0, 0]$ is $[1, -1, -1, 1, 1, -1, 1, 1, 1, 0, 0]$.

Theorem 1 ([YHZ96, Yang99]). *Given a polynomial* $f(x)$ *with real coefficients,* $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$*, if the number of sign changes of the revised sign list of* $[D_1(f), D_2(f), \cdots, D_n(f)]$ *is* v*, then the number of distinct pairs of conjugate imaginary roots of* $f(x)$ *equals* v*. Furthermore, if the number of non-vanishing members of the revised sign list is* l*, then the number of distinct real roots of* $f(x)$ *equals* $l - 2v$*.*

Definition 5. *Let* $M = \text{Discr}(f)$*. Denote by* M_k *the submatrix formed by the first* $2k$ *rows of* M*, for* $k = 1, \cdots, n$*; and* $M(k, i)$ *denotes the submatrix formed by the first* $2k-1$ *columns and the* $(2k+i)$*-th column of* M_k*, for* $k = 1, \cdots, n, i = 0, \cdots, n-k$*. Then, construct polynomials*

$$\Delta_k(f) = \sum_{i=0}^{k} \det(M(n-k, i))x^{k-i},$$

for $k = 0, 1, \cdots, n-1$*, where* $\det(M)$ *stands for the determinant of the square matrix* M*. We call the* n*-tuple*

$$\{\Delta_0(f), \Delta_1(f), \cdots, \Delta_{n-1}(f)\}$$

the multiple factor sequence *of* $f(x)$*.*

Lemma 1. *If the number of the* 0*'s in the revised sign list of the discrimination sequence of* $f(x)$ *is* k*, then* $\Delta_k(f) = gcd(f(x), f'(x))$*, i.e. the greatest common divisor of* $f(x)$ *and* $f'(x)$*.*

Definition 6. *By* $\mathcal{U}$ *denote the set of* $\{\text{gcd}^0(f), \text{gcd}^1(f), \cdots, \text{gcd}^k(f)\}$*, where* $\text{gcd}^0(f) = f$*,* $\text{gcd}^{i+1}(f) = gcd\,(\text{gcd}^i(f), \frac{\partial}{\partial x}\text{gcd}^i(f))$ *and* $\text{gcd}^k(f) = 1$*, i.e., all the greatest common divisors at different levels. Each polynomial in* $\mathcal{U}$ *has a discriminant sequence, and all of the discriminant sequences are called a* complete discrimination system *(CDS) of* $f(x)$*.*

Theorem 2 ([Yang99]). *If* $\mathrm{gcd}^j(f)$ *has* k *real roots with multiplicities* $n_1, n_2, ..., n_k$ *and* $\mathrm{gcd}^{j-1}(f)$ *has* m *distinct real roots, then* $\mathrm{gcd}^{j-1}(f)$ *has* k *real roots with multiplicities* $n_1+1, n_2+1, \cdots, n_k+1$ *and* $m-k$ *simple real roots.*

And the same argument is applicable to the imaginary roots.

Example 1. Let $f(x) = x^{18} - x^{16} + 2x^{15} - x^{14} - x^5 + x^4 + x^3 - 3x^2 + 3x - 1$. The sign list of the discrimination sequence of $f(x)$ is

$$[1, 1, -1, -1, -1, 0, 0, 0, -1, 1, 1, -1, -1, 1, -1, -1, 0, 0].$$

Hence, the revised sign list is

$$[1, 1, -1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, -1, -1, 0, 0],$$

of which the number of sign changes is seven, so $f(x)$ has seven pairs of distinct conjugate imaginary roots. Moveover, it has two distinct real roots and two repeated roots. Since $gcd(f, f') = x^2 - x + 1$, we know that f has two distinct real roots, one pair of conjugate imaginary roots with multiplicity 2 and six pairs of conjugate imaginary roots with multiplicity 1.

Theorem 3 ([Yang99, YX00]). *Let* $[d_1, d_2, \cdots, d_{2n+1}]$ *be the principal minor sequence of the discrimination matrix of the following polynomial*

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n \ (a_0 \neq 0,\ a_n \neq 0).$$

1. *Denote the number of sign changes and the number of non-vanishing members of the revised sign list of* n.r.d.(f), $[d_1 d_2, d_2 d_3, \cdots, d_{2n} d_{2n+1}]$, *by* v *and* $2l$, *respectively. Then, the number of distinct negative roots of* $f(x)$ *equals* $l - v$*;*
2. *Denote* $[d_2, d_4, ..., d_{2n}]$, $[d_1, d_3, ..., d_{2n+1}]$ *and* $[d_1 d_2, d_2 d_3, ..., d_{2n} d_{2n+1}]$ *by* L_1, L_2 *and* L_3*, respectively. If we denote the numbers of non-vanishing members and the numbers of sign changes of the revised sign lists of* $L_i\ (1 \leq i \leq 3)$ *by* l_i *and* v_i*, respectively, then* $l_3 = l_1 + l_2 - 1,\ v_3 = v_1 + v_2$.
3. *If* $d_{2m-1} = d_{2m+1} = 0$ *for some* $m\ (1 \leq m \leq n)$*, then* $d_{2m} = 0$.

Eliminating the quantifier in the formula

$$(\forall x > 0)\ (f(x) > 0) \tag{3}$$

is equivalent to finding the necessary and sufficient condition for $f(x)$ not having positive zeros. Similarly,

$$(\forall x \geq 0)\ (f(x) \geq 0) \tag{4}$$

is equivalent to the necessary and sufficient condition such that $f(x)$ does not have non-negative zeros or the non-negative zeros of $f(x)$ (if any) are all of even multiplicities. On the other hand, Theorems 1, 2 and 3 imply that, for a given polynomial $f(x)$, those conditions can be obtained by discussing on the signs of elements in the *negative root discriminant sequences* of $f(x)$ and $\mathrm{gcd}^i(f)$. Thus, a sketch of an algorithm for solving (3) can be described as follows which is similar to the combinatorial algorithm in [Gon98].

Algorithm: `Def-Con`

Input: A polynomial $f(x)$ with degree n and $f(0) \neq 0$

Output: The condition on the coefficients of $f(x)$ such that (3) holds

Step 1. Let $g(x) = f(-x)$ and denote by $[d_1, ..., d_{2n+1}]$ the list of principal minors of Discr (g).

Step 2. Discuss on all the possibilities of the signs of d_{2i}. Output those sign lists such that $l_1 - 2v_1 = 0$ (*i.e.*, $g(x)$ has no real zeros by Theorem 1) where v_1 and l_1 are the numbers of sign changes and non-vanishing members of the revised sign lists.

Step 3. For each list $[d_2, ..., d_{2n}]$ which makes $g(x)$ have real zeros, discuss on all the possibilities of the signs of d_{2i+1}. Output those sign lists of $[d_1, d_2, ..., d_{2n+1}]$ such that $l/2 - v = 0$ (*i.e.*, $g(x)$ has no negative zeros by Theorem 3) where v and l are the numbers of sign changes and non-vanishing members of the revised sign lists of n.r.d.(f).

Analogously, we may have an algorithm, named `Semi-Def-Con`, for solving (4) which is a little bit complicated since we have to use Theorem 2 to discuss on multiple zeros. In order to simplify the description, we suppose the first 3 steps in `Semi-Def-Con` are the same as those in `Def-Con`. So, we only need to consider those sign lists which make $g(x)$ have negative zeros and multiple zeros at the same time. For this case, we replace $f(x)$ by $\text{gcd}^i(f)$ with a suitable i, and run the first 3 steps recursively. By Theorem 2, we can get the condition for the negative zeros of $g(x)$ being all of even multiplicities.

By `Def-Con` and `Semi-Def-Con`, we can solve Problems (1) and (2) automatically. However, the results are much more complicated than those we shall give in the next section.

3 Main Results

Proposition 1. *Given a quartic polynomial of real coefficients,*

$$Q(\lambda) = \lambda^4 + p\,\lambda^3 + q\,\lambda^2 + r\lambda + s,$$

with $s \neq 0$*, then*

$$(\forall \lambda > 0)\; Q(\lambda) > 0$$

is equivalent to

$$\begin{aligned} s > 0 \wedge (&(p \geq 0 \wedge q \geq 0 \wedge r \geq 0) \vee \\ &(d_8 > 0 \wedge (d_6 \leq 0 \vee d_4 \leq 0)) \vee \\ &(d_8 < 0 \wedge d_7 \geq 0 \wedge (p \geq 0 \vee d_5 < 0)) \vee \\ &(d_8 < 0 \wedge d_7 < 0 \wedge p > 0 \wedge d_5 > 0) \vee \\ &(d_8 = 0 \wedge d_6 < 0 \wedge d_7 > 0 \wedge (p \geq 0 \vee d_5 < 0)) \vee \\ &(d_8 = 0 \wedge d_6 = 0 \wedge d_4 < 0)) \end{aligned} \tag{5}$$

where

$$
\begin{aligned}
d_4 &= -8\,q + 3\,p^2,\\
d_5 &= 3\,r\,p + q\,p^2 - 4\,q^2,\\
d_6 &= 14\,q\,r\,p - 4\,q^3 + 16\,s\,q - 3\,p^3\,r + p^2\,q^2 - 6\,p^2\,s - 18\,r^2,\\
d_7 &= 7\,r\,p^2\,s - 18\,q\,p\,r^2 - 3\,q\,p^3\,s - q^2\,p^2\,r + 16\,s^2\,p + 4\,r^2\,p^3 + 12\,q^2\,p\,s\\
&\quad + 4\,r\,q^3 - 48\,r\,s\,q + 27\,r^3,\\
d_8 &= p^2\,q^2\,r^2 + 144\,q\,s\,r^2 - 192\,r\,s^2\,p + 144\,q\,s^2\,p^2 - 4\,p^2\,q^3\,s + 18\,q\,r^3\,p\\
&\quad - 6\,p^2\,s\,r^2 - 80\,r\,p\,s\,q^2 + 18\,p^3\,r\,s\,q - 4\,q^3\,r^2 + 16\,q^4\,s - 128\,s^2\,q^2\\
&\quad - 4\,p^3\,r^3 - 27\,p^4\,s^2 - 27\,r^4 + 256\,s^3.
\end{aligned}
$$

Proof. We need to find the necessary and sufficient condition such that $Q(\lambda)$ does not have positive zeros. First of all, by Cartesian sign rule we have the following results:

1. $s > 0$ must hold. Otherwise, the sequence $[1, p, q, r, s]$ will have an odd number of sign changes which implies $Q(\lambda)$ has at least one positive zero.
2. If the zeros of $Q(\lambda)$ are all real, $Q(\lambda)$ does not have positive zeros if and only if $s > 0$ and p, q, r are all non-negative.

Therefore, in the following we always assume $s > 0$ and do not consider the case when $Q(\lambda)$ has four real zeros (counting multiplicity).

Let $P(\lambda) = Q(-\lambda)$, then we discuss the condition such that $P(\lambda)$ does not have negative zeros. We compute the principal minors d_i $(1 \le i \le 9)$ of $\mathrm{Discr}(P)$ and consider the following two lists:

$$L_1 = [1, d_4, d_6, d_8] \;\text{ and }\; L_2 = [1, d_3, d_5, d_7, d_9]$$

where $d_3 = -p,\ d_9 = s d_8$ and d_i $(4 \le i \le 8)$ are showed above in the statement of this proposition. In the following, we denote the numbers of non-vanishing elements and sign changes of the revised sign list of L_i by l_i and v_i $(i = 1, 2)$, respectively.

Case **I.** $d_8 > 0$.

In this case, by Theorem 1 $P(\lambda)$ has either four imaginary zeros or four real zeros. $P(\lambda)$ has four imaginary zeros if and only if $d_6 \le 0 \vee d_4 \le 0$ by Theorem 1. As stated above, we need not to consider the case when $P(\lambda)$ has four real zeros. Thus,

$$d_8 > 0 \;\wedge\; (d_6 \le 0 \;\vee\; d_4 \le 0)$$

must be satisfied under Case I.

Case **II.** $d_8 < 0$.

In this case, L_1 becomes $[1, d_4, d_6, -1]$ with $l_1 = 4, v_1 = 1$ which implies by Theorem 1 that $P(\lambda)$ has two imaginary zeros and two distinct real zeros.

If $d_7 > 0$, L_2 becomes $[1, -p, d_5, 1, -1]$. By Theorem 3, v_2 should be 3 which is equivalent to $p \ge 0 \;\vee\; d_5 \le 0$.

If $d_7 = 0$, L_2 becomes $[1, -p, d_5, 0, -1]$. By Theorem 3, v_2 should be 3 which is equivalent to $p \ge 0 \;\vee\; d_5 < 0$.

To combine the above two conditions, we perform pseudo-division of d_7 and d_5 with respect to r and obtain that

$$27p^3 d_7 = F d_5 + 12G^2 \tag{6}$$

where F, G are polynomials in p, q, r, s. It's easy to see that p should be non-negative if $d_7 > 0$ and $d_5 = 0$. Thus, we may combine the above two sub-cases into

$$d_7 \geq 0 \ \wedge \ (p \geq 0 \ \vee \ d_5 < 0).$$

If $d_7 < 0$, L_2 becomes $[1, -p, d_5, -1, -1]$. By (6) we know that $p = 0 \wedge d_5 > 0$ and $p > 0 \wedge d_5 = 0$ are both impossible. Thus, v_2 is 3 if and only if $p > 0 \wedge d_5 > 0$.

In Case II, We conclude that

$$d_8 < 0 \wedge \ [(d_7 \geq 0 \wedge (p \geq 0 \vee d_5 < 0)) \vee (d_7 < 0 \wedge p > 0 \wedge d_5 > 0)]$$

must be satisfied.

Case **III.** $d_8 = 0$.

If $d_6 > 0$, $P(\lambda)$ has four real zeros (counting multiplicity) and this is the case having been discussed already.

If $d_6 < 0$, then $l_1 = 3$ and $v_1 = 1$. We need to find the condition for $l_2/2 = v_2$ by Theorem 3. Obviously, l_2 must be an even integer. We consider the sign of d_7. First, $d_7 < 0$ implies $l_2/2 = 2$ and v_2 is an odd integer and thus $l_2/2 = v_2$ can not be satisfied. Second, if $d_7 = 0$, by Theorem 3 $d_5 \neq 0$ since $d_6 < 0$. That means l_2 is odd which is impossible. Finally, if $d_7 > 0$, v_2 must be 2 and this is satisfied by $p \geq 0 \ \vee \ d_5 < 0$.

If $d_6 = 0$, L_1 becomes $[1, d_4, 0, 0]$. And $d_4 \geq 0$ implies $P(\lambda)$ has four real zeros (counting multiplicity) and this is the case having been discussed already. If $d_4 < 0$, $P(\lambda)$ has four imaginary zeros (counting multiplicity) and thus no negative zeros.

In Case III, we conclude that

$$d_8 = 0 \ \wedge \ [(d_6 < 0 \wedge d_7 > 0 \wedge (p \geq 0 \vee d_5 < 0)) \ \vee \ (d_6 = 0 \wedge d_4 < 0)]$$

must be satisfied. That completes the proof. □

Proposition 2. *Given a quartic polynomial of real coefficients,*

$$Q(\lambda) = \lambda^4 + p\,\lambda^3 + q\,\lambda^2 + r\lambda + s,$$

with $s \neq 0$*, then*

$$(\,\forall\, \lambda \geq 0\,) \quad Q(\lambda) \geq 0$$

is equivalent to

$$\begin{aligned} s > 0 \wedge ((p \geq 0 \wedge q \geq 0 \wedge r \geq 0) \vee &\\ (d_8 > 0 \wedge (d_6 \leq 0 \vee d_4 \leq 0)) \vee &\\ (d_8 < 0 \wedge d_7 \geq 0 \wedge (p \geq 0 \vee d_5 < 0)) \vee &\\ (d_8 < 0 \wedge d_7 < 0 \wedge p > 0 \wedge d_5 > 0) \vee &\\ (d_8 = 0 \wedge d_6 < 0) \vee &\\ (d_8 = 0 \wedge d_6 > 0 \wedge d_7 > 0 \wedge (p \geq 0 \vee d_5 < 0)) \vee &\\ (d_8 = 0 \wedge d_6 = 0 \wedge (d_4 \leq 0 \vee E_1 = 0))) & \end{aligned} \tag{7}$$

where d_i $(4 \leq i \leq 8)$ *are defined as in Proposition 1 and*

$$E_1 = 8\,r - 4\,p\,q + p^3.$$

Proof. Because $Q(0) = s \neq 0$, it is equivalent to consider

$$(\forall\, \lambda > 0\,) \quad Q(\lambda) \geq 0.$$

And the formula holds if and only if $Q(\lambda)$ has no positive zeros or each positive zero (if any) of $Q(\lambda)$ is of even multiplicity.

Since the first case that $Q(\lambda)$ has no positive zeros has been discussed in Proposition 1, we only discuss on the later case. So, we assume that $s > 0$ and $d_8 = 0$. All notations are as in Proposition 1.

Case **I.** $d_6 < 0$.

L_1 becomes $[1, d_4, -1, 0]$ which implies $P(\lambda)$ has a pair of imaginary zeros and one real zero with multiplicity 2. Thus, $P(\lambda)$ is positive semi-definite no matter what value λ is.

Case **II.** $d_6 > 0$.

In this case, L_1 becomes $[1, d_4, 1, 0]$ which implies that $P(\lambda)$ has three distinct real zeros of which one is of multiplicity 2. By Cartesian sign rule, the number of positive real zeros (counting multiplicity) is even. Therefore, we need only to find the condition such that $Q(\lambda)$ has one distinct positive real zero (*i.e.*, $P(\lambda)$ has one distinct negative real zero). Because $l_1 = 3,\ v_1 = 0$, by Theorem 3, it must be $l_2/2 = v_2 = 2$. And this is true if and only if $d_7 > 0 \wedge\ (p \geq 0 \vee d_5 \leq 0)$. From (6), we know that $d_7 > 0 \wedge p < 0 \wedge d_5 = 0$ is impossible. So we conclude that, in this case, the following formula should be true.

$$d_6 > 0 \wedge\ d_7 > 0 \wedge\ (p \geq 0 \vee d_5 < 0).$$

Case **III.** $d_6 = 0$.

Since the case that $d_4 \leq 0$ has been discussed, we assume $d_4 > 0$ which implies that $P(\lambda)$ has two distinct real zeros and no imaginary zeros. Because the case that $P(\lambda)$ has no negative zeros has been discussed as stated above, we must find the condition such that each of the two real zeros of $P(\lambda)$ is of multiplicity 2.

We can obtain the condition by discussing on the root classification of the repeated part of $P(\lambda)$ through Theorem 2. But the condition obtained is a little bit complex than the one obtained in the following way.

Suppose $Q = (\lambda^2 + a\lambda + b)^2$, we get

$$(-p + 2a)\lambda^3 + (2b + a^2 - q)\lambda^2 + (2ab - r)\lambda + b^2 - s = 0.$$

So

$$-p + 2a = 0, 2b + a^2 - q = 0, 2ab - r = 0, b^2 - s = 0, \tag{8}$$

where a, b are indeterminates. Substituting $p/2$ for a in the equations, we get $2b+1/4p^2-q = 0, -r+pb = 0, b^2-s = 0$. Suppose $p \neq 0$ and substituting $b = r/p$ into the equalities, we get $E_1 = 0$ and $E_2 = 0$ where $E_1 = 8r - 4pq + p^3,\ E_2 = r^2 - p^2 s$.

If $p = 0, E_1 = 0$ and $E_2 = 0$, then $r = 0, d_4 = -8q, d_6 = 4q(4s - q^2)$. Under the precondition that $d_6 = 0 \wedge d_4 > 0$, we have $4s - q^2 = 0$ which solves equations (8) together with $p = r = 0$. In a word, the equations (8) has common solutions if and only if $E_1 = 0$ and $E_2 = 0$ under the precondition that $d_6 = 0 \wedge d_4 > 0$.

On the other hand, we have

$$p^2 d_6 = 2d_4E_2 + (2rq - 3rp^2 + pq^2)E_1.$$

If $d_6 = 0$ and $d4 > 0$, $E_2 = 0$ is implied by $E_1 = 0$. Finally, we conclude in this case that

$$d_6 = 0 \wedge d_4 > 0 \wedge E_1 = 0$$

should be true.

That ends the proof. □

Remark 1. We have tried the two problems by our Maple program DISCOVERER [YHX01, YX05] which includes an implementation of the algorithms in Section 2 and obtained some quantifier-free formulas equivalent to those of (5) and (7). However, the formulas are much more complicated than the ones stated in Propositions 1 and 2. For example, the resulting formula for Problem (1) are as follows.

$$
\begin{aligned}
s > 0 \wedge [\ & [d_8 < 0, d_7 <= 0, d_6 < 0, 0 < d_5, d_4 <> 0, d_3 < 0] \vee \\
& [d_8 <= 0, 0 < d_7, d_6 < 0, d_5 < 0] \vee \\
& [d_8 <= 0, 0 < d_7, d_6 < 0, 0 <= d_5, d_4 <> 0, d_3 < 0] \vee \\
& [d_8 < 0, d_6 < 0, 0 < d_5, d_4 = 0, d_3 < 0] \vee \\
& [d_8 < 0, 0 < d_7, d_6 <= 0, d_5 = 0, d_4 = 0, d_3 <= 0] \vee \\
& [d_8 < 0, d_7 < 0, d_6 = 0, d_5 < 0, d_4 = 0, d_3 <= 0] \vee \\
& [d_8 < 0, d_7 < 0, d_6 = 0, d_5 = 0, d_4 = 0, d_3 = 0] \vee \\
& [d_8 < 0, d_7 < 0, d_6 = 0, 0 < d_5, 0 <= d_4, d_3 < 0] \vee \\
& [d_8 < 0, d_7 = 0, d_6 = 0, d_5 = 0, d_4 = 0] \vee \\
& [d_8 < 0, 0 <= d_7, d_6 = 0, 0 < d_5, d_4 = 0, d_3 <> 0] \vee \\
& [d_8 < 0, 0 < d_7, d_6 = 0, d_5 < 0, 0 <= d_4] \vee \\
& [d_8 < 0, 0 <= d_7, d_6 = 0, 0 <= d_5, 0 < d_4, d_3 < 0] \vee \\
& [d_8 < 0, d_7 <= 0, 0 < d_6, 0 < d_5, 0 < d_4, d_3 < 0] \vee
\end{aligned}
$$

$$[d_8 < 0, 0 < d_7, 0 < d_6, d_5 < 0, 0 < d_4] \vee$$
$$[d_8 < 0, 0 < d_7, 0 < d_6, 0 <= d_5, 0 < d_4, d_3 < 0] \vee$$
$$[d_8 = 0, 0 < d_7, d_6 < 0, d_5 = 0, d_4 = 0, d_3 <= 0] \vee$$
$$[d_8 = 0, 0 < d_7, d_6 < 0, 0 < d_5, d_4 = 0, d_3 < 0] \vee$$
$$[d_8 = 0, d_6 = 0, d_4 < 0] \vee$$
$$[d_8 = 0, d_7 < 0, d_6 = 0, d_5 <> 0, d_4 = 0, d_3 < 0] \vee$$
$$[d_8 = 0, d_7 = 0, d_6 = 0, d_4 = 0, d_3 < 0] \vee$$
$$[d_8 = 0, d_6 = 0, d_4 = 0, d_3 < 0] \vee$$
$$[d_8 = 0, d_6 = 0, 0 < d_5, 0 < d_4, d_3 < 0] \vee$$
$$[0 <= d_8, d_7 < 0, 0 < d_6, 0 < d_5, 0 < d_4, d_3 < 0] \vee$$
$$[0 < d_8, d_6 <= 0] \vee$$
$$[0 < d_8, 0 < d_6, d_4 <= 0]\ ]$$

Here, $d_3 = -p$ and the other d_is are defined as in Proposition 1. The above formula contains much more clauses than formula (5). For Problem (2), the resulting formula created by DISCOVERER is even more complicated because we have to add some more clauses for the cases existing positive zeros with even multiplicities.

Remark 2. We use Cartesian sign rule in the proofs of Propositions 1 and 2. This can be integrated into `Def-Con` and `Semi-Def-Con` to produce simpler formulas. In fact, a naive use of Cartesian sign rule may decrease the number of clauses. Some optimal strategy on sign discussion and result simplification can also be implemented. However, some computation like pseudo-division in the proofs depends on each concrete problem and thus is hard to be turned into an algorithm.

4 Two Examples in Application

Our first example comes from determination of termination of linear loop programs. Termination analysis plays a central role in formal verification of programs [Cou00]. An ideal solution to the termination problem for a class of programs is to prove the decidability of its termination problem and to establish calculable conditions so that for any given specific program in the class, we can compute these conditions to conclude whether the given program terminates.

The linear programs [BJT99, CH78, HPR97] are a class of programs that is widely studied. A large number of reactive systems can be modelled precisely or approximately as the linear programs [HH95]. Unfortunately, the termination problem of linear programs is undecidable in general [Tiw04]. However, Tiwari proves [Tiw04] the decidability of a specific class of linear loop programs of the form

$$\mathbf{P_1}: \ \textbf{while} \ \ Bx > b \ \{x := Ax + c\}$$

where x (b and c) is a vector of N program variables (and real numbers), A and B are $N \times N$ and $N \times M$ real matrices respectively, $Bx > b$ represents a conjunction of M linear inequalities in the program variables and $x := Ax + c$ represents the linear assignments to each of the variables.

Theorem 4 ([Tiw04]). *The termination of nonhomogeneous linear program of* $\mathbf{P_1}$ *is decidable.*

Denote the homogeneous case of the program $\mathbf{P}_1$ where b and c both are 0 by

$$\mathbf{P}_2: \quad \textbf{while} \ (Bx > 0) \ \{x := Ax\}.$$

Theorem 5 ([Tiw04]). *If the program $\mathbf{P_2}$ is nonterminating, then there is a real eigenvector v of A, corresponding to a positive real eigenvalue, such that $Bv \geq 0$.*

Definition 7. *Assignment $x := Ax$ of $\mathbf{P_2}$ is called a* terminating assignment*, if matrix A has no positive eigenvalue.*

Obviously, if $x := Ax$ of $\mathbf{P_2}$ is a terminating assignment, then $\mathbf{P_2}$ terminates for any matrix B. By the above definition, we have the following theorem as a direct result from Proposition 1. The theorem first appeared in [YZXZ05] without proof due to page limitation.

Theorem 6. *Suppose A is a 4×4 matrix*

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix},$$

$x := Ax$ is a terminating assignment if and only if the condition (5) is satisfied where

$$\begin{aligned}
p &= -a_{11} - a_{22} - a_{33} - a_{44} \\
q &= a_{33}a_{44} + a_{11}a_{22} - a_{41}a_{14} - a_{31}a_{13} - a_{32}a_{23} - a_{34}a_{43} + a_{22}a_{44} + a_{22}a_{33} \\
&\quad -a_{21}a_{12} - a_{42}a_{24} + a_{11}a_{44} + a_{11}a_{33} \\
r &= -a_{32}a_{24}a_{43} + a_{11}a_{34}a_{43} - a_{11}a_{33}a_{44} - a_{21}a_{42}a_{14} + a_{11}a_{32}a_{23} + a_{21}a_{12}a_{33} + \\
&\quad a_{42}a_{24}a_{33} + a_{11}a_{42}a_{24} - a_{31}a_{12}a_{23} + a_{22}a_{34}a_{43} - a_{11}a_{22}a_{33} + a_{31}a_{13}a_{44} - \\
&\quad a_{11}a_{22}a_{44} - a_{42}a_{23}a_{34} - a_{22}a_{33}a_{44} - a_{41}a_{12}a_{24} + a_{32}a_{23}a_{44} - a_{41}a_{13}a_{34} + \\
&\quad a_{41}a_{14}a_{33} + a_{21}a_{12}a_{44} + a_{41}a_{22}a_{14} - a_{31}a_{14}a_{43} + a_{31}a_{22}a_{13} - a_{21}a_{32}a_{13} \\
s &= -a_{11}a_{22}a_{34}a_{43} - a_{21}a_{32}a_{14}a_{43} - a_{21}a_{42}a_{13}a_{34} + a_{11}a_{32}a_{24}a_{43} + \\
&\quad a_{21}a_{42}a_{14}a_{33} + a_{41}a_{12}a_{24}a_{33} + a_{31}a_{12}a_{23}a_{44} - a_{31}a_{12}a_{24}a_{43} + \\
&\quad a_{11}a_{22}a_{33}a_{44} - a_{21}a_{12}a_{33}a_{44} + a_{21}a_{12}a_{34}a_{43} - a_{31}a_{22}a_{13}a_{44} - \\
&\quad a_{41}a_{12}a_{23}a_{34} + a_{31}a_{22}a_{14}a_{43} - a_{31}a_{42}a_{14}a_{23} - a_{11}a_{32}a_{23}a_{44} + \\
&\quad a_{41}a_{22}a_{13}a_{34} + a_{11}a_{42}a_{23}a_{34} - a_{11}a_{42}a_{24}a_{33} + a_{41}a_{32}a_{14}a_{23} + \\
&\quad a_{21}a_{32}a_{13}a_{44} - a_{41}a_{22}a_{14}a_{33} - a_{41}a_{32}a_{13}a_{24} + a_{31}a_{42}a_{13}a_{24}
\end{aligned}$$

Our second example comes from the determination of positivity of symmetric polynomials with degree 4 and arbitrary number of variables. Let $\mathbb{R}$ be the real numbers, $\mathbb{R}^n_+ = \{(x_1, ..., x_n) | x_i \in \mathbb{R}, x_i \geq 0\}$, $1_k = (\overbrace{1, ..., 1}^{k})$, $0_k = (\overbrace{0, ..., 0}^{k})$ and $H_d^{[n]}$ the set of real symmetric d-homogeneous polynomials in n variables. For any $x = (x_1, ..., x_n) \in \mathbb{R}^n$, set

$$v(x) = |\{x_i|\ i = 1, ..., n\}|,\ v^*(x) = |\{x_i|\ x_i \neq 0,\ i = 1, ..., n\}|.$$

That is to say, $v(x)$ is the number of distinct elements in x and $v^*(x)$ is the number of distinct non-zero elements in x. V. Timofte proves the following result.

Theorem 7 ([Tim03]). *Suppose $f(x) \in H_d^{[n]}$. Then $f \geq 0$ holds on $\mathbb{R}_+^n$ if and only if it holds for $x \in \{x|\ x \in \mathbb{R}_+^n,\ v^*(x) \leq \max([\frac{d}{2}], 1)\}$.*

Set $N_n = \{(r,s)|\ r,s$ are positive integers with $r+s \leq n\}$. If $d = 4$, it's easy to see that

$$\begin{aligned} & f(x) \geq 0,\ x \in \mathbb{R}_+^n \\ \Longleftrightarrow\ & f(x) \geq 0,\ x \in \mathbb{R}_+^n,\ v^*(x) \leq 2 \\ \Longleftrightarrow\ & f(t_1 \cdot 1_r, t_2 \cdot 1_s, 0_{n-r-s}) \geq 0,\ \forall t_1, t_2 \geq 0,\ \forall (r,s) \in N_n \\ \Longleftrightarrow\ & t_2^a f(\tfrac{t_1}{t_2} \cdot 1_r, 1_s, 0_{n-r-s}) \geq 0,\ \forall t_1 \geq 0, \forall t_2 > 0,\ \forall (r,s) \in N_n \\ \Longleftrightarrow\ & f(t \cdot 1_r, 1_s, 0_{n-r-s}) \geq 0,\ \forall t \geq 0,\ \forall (r,s) \in N_n \end{aligned}$$

Therefore, to determine the positivity of a polynomial in $H_4^{[n]}$ on $\mathbb{R}_+^n$, it's sufficient to determine the positivity of a finite number of polynomials in one non-negative variable with degree 4. The result of Proposition 2 is exactly suitable for the determination and if n is very large, the determination will benifit from such off-line condition as (7).

Acknowledgments

The authors would like to thank the anonymous referees for their valuable comments on a previous version of this paper. The work is supported by NKBRPC-2004CB318003, NKBRPC-2005CB321902 and NSFC-60573007 in China.

References

[AH00] H. Anai, S. Hara: Fixed-structure robust controller synthesis based on sign definite condition by a special quantifier elimination. In: *Proc. American Control Conference 2000*, pp. 1312–1316, 2000.

[ACM84a] D. S. Arnon, G. E. Collins and S. McCallum: Cylindrical algebraic decomposition I: The basic algorithm. *SIAM J. Comput.* **13** (1984): 865–877.

[ACM84b] D. S. Arnon, G. E. Collins and S. McCallum: Cylindrical algebraic decomposition II: An adjacency algorithm for the plane. *SIAM J. Comput.* **13** (1984): 878–889.

[ACM88] D. S. Arnon, G. E. Collins and S. McCallum: Cylindrical algebraic decomposition III: An adjacency algorithm for three-dimensional space. *J. Symb. Comput.*: 163–187 (1988).

[AM88] D. S. Arnon and M. Mignotte: On mechanical quantifier elimination for elementary algebra and geometry. *J. Symb. Comput.*, 5:237–260, 1988.

[BJT99] F. Besson, T. Jensen, and J.-P. Talpin: Polyhedral analysis of synchronous languages. In SAS'99, LNCS 1694, pp. 51–69, Springer-Verlag, 1999.

[Br01a] C. W. Brown: Simple CAD construction and its applications. *J. Symb. Comput.* **31** (2001): 521–547.

[Br01b] C. W. Brown: Improved projection for cylindrical algebraic decomposition. *J. Symb. Comput.* **32** (2001): 447–465.

[BM05] C. W. Brown and S. McCallum: On Using Bi-equational Constraints in CAD Construction. In: *Proc. ISSAC2005* (Kauers, M. ed.), 76–83, ACM Press, New York (2005).

[CJ98] B. F. Caviness and J. R. Johnson(eds.), *Quantifier Elimination and Cylindrical Algebraic Decomposition*, Springer-Verlag, 1998.

[Co75] G. E. Collins: Quantifier elimination for real closed fields by cylindrical algebraic decomposition. In: *Lecture Notes in Computer Science* **33**, pp. 134–165. Springer-Verlag, Berlin Heidelberg (1975).

[Co98] G. E. Collins: Quantifier elimination by cylindrical algebraic decomposition - 20 years of progress. In: *Quantifier Elimination and Cylindrical Algebraic Decomposition* (Caviness, B. and Johnson, J. eds.), 8–23. Springer-Verlag, New York (1998).

[CH91] G. E. Collins and H. Hong: Partial cylindrical algebraic decomposition for quantifier elimination. *J. Symb. Comput.* **12**: 299–328 (1991).

[Cou00] P. Cousot: Abstract interpretation based formal methods and future challenges. In *Informatics, 10 Years Back - 10 Years Ahead*, R. Wilhelm (Ed.), LNCS 2000, pp. 138–156, 2001.

[CH78] P. Cousot and N. Halbwachs: Automatic discovery of linear restraints among the variables of a program. In ACM POPL'78, pp. 84-97, 1978.

[DSW98] A. Dolzmann, Th. Sturm and V. Weispfenning: Real quantifier elimination in practice, *Algorithmic Algebra and Number Theory*, Matzat, B. H., Greuel, G.-M. and Hiss, G. (eds.), Springer 1998, pp. 221–247.

[Ga59] F. R. Gantmacher: *The Theory of Matrices*. Chelsea Publishing Company, New York, 1959.

[Gon98] L. González-Vega: A Combinatorial Algorithm Solving Some Quantifier Elimination Problems. In: *Quantifier Elimination and Cylindrical Algebraic Decomposition* (Caviness, B. and Johnson, J. eds.), 365–375. Springer-Verlag, New York (1998).

[GLRR89] L. González-Vega, H. Lombardi, T. Recio, M.-F. Roy: Sturm-Habicht sequence. In *Proc. of ISSAC'89*, ACM Press, pp. 136–146(1989).

[HPR97] N. Halbwachs, Y.E. Proy, and P. Roumanoff: Verification of real-time systems using linear relation analysis. *Formal Methods in System Design,* 11(2):157–185, 1997.

[HH95] T.A. Henzinger and P.-H. Ho: Algorithmic analysis of nonlinear hybrid systems. In CAV'95, LNCS 939, pp.225–238. 1995.

[Hong90] H. Hong: An improvement of the projection operator in cylindrical algebraic decomposition. In: *Proceedings of ISSAC '90* (Watanabe, S. and Nagata, M. eds.), pp. 261–264. ACM Press, New York (1990).

[Hong92] H. Hong: Simple solution formula construction in cylindrical algebraic decomposition based quantifier elimination. In: *Proceedings of ISSAC '92* (Wang, P. S., ed.), pp. 177–188. ACM Press, New York (1992).

[Hong96] H. Hong: An efficient method for analyzing the topology of plane real algebraic curves. *Math. Comput. Simul.* **42** (1996): 571–582.

[La88] D. Lazard: Quantifier elimination: optimal solution for two classical examples. *J. Symb. Comput.* 5:261–266, 1988.

[Mc88] S. McCallum: An improved projection operation for cylindrical algebraic decomposition of three-dimensional space. *J. Symb. Comput.*: 141–161(1988).

[Mc98] S. McCallum: An improved projection operator for cylindrical algebraic decomposition. In: *Quantifier Elimination and Cylindrical Algebraic Decomposition* (Caviness, B. and Johnson, J. eds.), 242–268. Springer-Verlag, New York (1998).

[Re92] J. Renegar: On the Computational Complexity and Geometry of the First-order Theory of the Reals. Parts I, II, and III. *J. Symb. Comput.*, **13**, pp. 255–352, 1992.

[Ta51] A. Tarski: *A Decision for Elementary Algebra and Geometry.* University of California Press, Berkeley, May. 1951.

[Tim03] V. Timofte: On the positivity of symmetric polynomial functions. Part I: General results. *J. Math. Anal. Appl.*, **284**, pp. 174–190, 2003.

[Tiw04] A. Tiwari: Termination of linear programs. In CAV'04, LNCS 3114, pp. 70–82, 2004.

[WY00] L. Wang and W. Yu: Complete characterization of strictly positive real regions and robust strictly positive real synthesis method, *Science in China*, **(E)43**:97-112, 2000.

[WH99] Z. H. Wang and H. Y. Hu: Delay-independent stability of retarded dynamic systems of multiple degrees of freedom, *Journal of Sound and Vibration*, **226**(1), 57-81, 1999.

[WH00] Z. H. Wang and H. Y. Hu: Stability of time-delayed dynamic systems with unknown parameters, *Journal of Sound and Vibration* **233**(2), 215-233, 2000.

[Wei94] V. Weispfenning: Quantifier elimination for real algebra - the cubic case. In *Proc. ISSAC 94*, Oxford, 1994, ACM Press, pp. 258–263.

[Wei97] V. Weispfenning: Quantifier elimination for real algebra - the quadratic case and beyond. *AAECC 8*, 1997, pp. 85–101.

[Wei98] V. Weispfenning: A New Approach to Quantifier Elimination for Real Algebra. In: *Quantifier Elimination and Cylindrical Algebraic Decomposition* (Caviness, B. and Johnson, J. eds.), 376–392. Springer-Verlag, New York (1998).

[Wu92] W.T. Wu: On problems involving inequalities. In: *MM-Preprints*, No.7, pp. 1–13, 1992.

[Yang99] L. Yang: Recent advances on determining the number of real roots of parametric polynomials. *J. Symbolic Computation*, 28:225–242, 1999.

[YHX01] L. Yang, X. Hou and B. Xia: A complete algorithm for automated discovering of a class of inequality-type theorems. *Sci. in China (Ser. F)* 44: 33–49 (2001).

[YHZ96] L. Yang, X. Hou and Z. Zeng: A complete discrimination system for polynomials. *Science in China (Ser. E)*, 39:628–646, 1996.

[YX00] L. Yang and B. Xia: An explicit criterion to determine the number of roots of a polynomial on an interval. *Progress in Natural Science*, 10(12):897–910, 2000.

[YX05] L. Yang and B. Xia: Real solution classifications of a class of parametric semi-algebraic systems. In *Proc. of Int'l Conf. on Algorithmic Algebra and Logic*, 281–289, 2005.

[YZXZ05] L. Yang, N. Zhan, B. Xia and C. Zhou: Program Verification by Using DISCOVERER. To appear in Proc. of the IFIP Working Conference on Verified Software: Tools, Techniques and Experiments, Zurich, 10-13 October 2005.