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Abstract

For a famous cubic system given by James and Lloyd, there exist some sufficient conditions such that the system has eight limit cycles. In this paper, we try to derive by computers the necessary and sufficient conditions for this system to have eight limit cycles. In order to find the symbolic real solutions to semi-algebraic systems where polynomials are Lyapunov quantities, we transform the equations into triangular systems by pseudo-division, locate the real solutions of the last equation and verify the inequalities by the Budan-Fourier theorem. The necessary and sufficient conditions for the system to have eight limit cycles are given under a reasonable limitation.

1 Introduction

Along with the rapid progress of computer hardware and software, especially the generalization of computer algebra systems, more and more problems from various fields of mathematics are solved by computers. In the research on the problem of maximum number of limit cycles for polynomial differential systems, which is a part of Hilbert's 16th problem, the use of computers and computer algebra systems is very impressive. There are many exciting results in this respect [8, 12, 6, 10, 9].

E. M. James and N. G. Lloyd construct a cubic system [6] which, to our knowledge, is by now the only cubic system that is proven by computer to have eight small-amplitude limit cycles [6, 10, 9]. The system is

$$\begin{aligned} \dot{x} &= y + a_1x^2 - 2b_1xy + (a_3 - a_1)y^2 + a_5x^2y + a_7y^3, \\ \dot{y} &= -x + b_1x^2 + 2a_1xy - b_1y^2 + b_4x^3 + b_5x^2y + (b_6 - a_5)xy^2. \end{aligned} \tag{1}$$

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Existing conditions [6, 10, 9] for the system (1) to have eight limit cycles are all sufficient ones. In this paper, we try to derive a necessary and sufficient condition. To simplify the complicated computations, we introduce some new parameters a_9, a_8 and b_8 as in [6] by letting $a_9 = a_7 - a_5 + b_6/2,\ a_8 = b_1^2 - a_1^2$ and $b_8 = b_1^2 + a_1^2$; thus, we should have $b_8 > 0$ and $b_8^2 - a_8^2 > 0$ if $a_1 \neq 0$ or $b_1 \neq 0$. To ensure the origin to be an eighth-order focus, we must have

$$L(1) = 0, ..., L(7) = 0, \quad \text{and} \quad L(8) \neq 0, \tag{2}$$

where $L(k)$ are Lyapunov quantities for $k = 1, 2, ...$ Now, the problem is to derive conditions for the following semi-algebraic system to have real solution(s).

$$\begin{cases} L(1) = 0, ..., L(7) = 0, \\ L(8) \neq 0, \\ b_8 > 0,\ b_8^2 - a_8^2 > 0. \end{cases} \tag{3}$$

This is a problem of quantifier elimination on real closed fields. Theoretically speaking, it can be decided by such famous algorithms as Tarski's method [11] and cylindrical algebraic decomposition [2] and other methods [4, 5, 15]. But, all those algorithms are not practical to our problem because of heavy computations. Ritt-Wu's zero structure decomposition method [13, 14] and the Gröbner basis method [1] are not practicable for the equations in the system (2) since the polynomials are very large. The Sturm theorem was employed in [6] to locate real roots of polynomial equation and to determine the sign of a polynomial on an interval. However, computer implementations of Sturm's theorem are generally too inefficient on large polynomials [3, 7].

We propose the following strategy for finding the symbolic real solutions to the system (3). First, we transform the equations into triangular systems by pseudo-division, and then locate the real solutions of the last equation and verify the inequalities by the Budan-Fourier theorem, which is much more efficient than Sturm's theorem though incomplete for the problems in general. Our method has been found to be computationally efficient in practice on this kind of problems.

With the above strategy for solving semi-algebraic systems, we obtain the necessary and sufficient conditions for the origin to be an eighth-order fine focus of the system (1) under a reasonable limitation. Those conditions are also the necessary and sufficient conditions for the system to have eight small-amplitude limit cycles under the same limitation.

2 Preliminaries

In this section, for convenience of the reader, we recall briefly the basic concepts and results concerning pseudo-division and Budan-Fourier's theorem.

Let $\mathbb{D}$ be a commutative ring, F a polynomial in $\mathbb{D}[x_1, ..., x_n]$ and x_k a fixed variable. While considered as a polynomial in x_k, F can be written as $F = F_0 x_k^p + F_1 x_k^{p-1} + \cdots + F_p$, where $F_i \in \mathbb{D}[x_1, ..., x_{k-1}, x_{k+1}, ..., x_n]$ and p is the degree of F in x_k and denoted by $\deg(F, x_k)$. F_0 is the *leading coefficient* of F in x_k, denoted by $\mathrm{lc}(F, x_k)$.

Let F and G be two polynomials in $\mathbb{D}[x_1, ..., x_n]$ and $G \neq 0$, $q = \deg(G, x_k)$, $p = \deg(F, x_k)$. For pseudo-dividing F by G, considered as polynomials in x_k, we have a division algorithm [13, 14] as follows. Let $R \leftarrow F$; repeat the following process until $r = \deg(R, x_k) < q$:

$$R \leftarrow G_0 R - R_0 x_k^{r-q} G,$$

where $G_0 = \mathrm{lc}(G, x_k)$, $R_0 = \mathrm{lc}(R, x_k)$. Finally, one obtains two polynomials Q and R in $\mathbb{D}[x_1, ..., x_n]$ satisfying the relation

$$I^s F = QG + R, \tag{4}$$

where $I = G_0$, $s = \max(p - q + 1, 0)$, $\deg(R, x_k) < q$. In case $q = 0$, $R = 0$ and $Q = G^p F$. Q and R are called the pseudo-quotient and pseudo-remainder of F divided by G, respectively.

Let $\mathrm{Zero}(F, G)$ denotes the set of all common zeros (in some field extension of $\mathbb{D}$) of F and G, $\mathrm{Zero}(G, R/I) = \mathrm{Zero}(G, R) \setminus \mathrm{Zero}(I)$, then

$$\mathrm{Zero}(G, R) \supseteq \mathrm{Zero}(F, G) \supseteq \mathrm{Zero}(G, R/I).$$

Budan-Fourier's Theorem *Suppose $f(x) = 0$ is a polynomial equation of degree n with real coefficients, a and b ($a < b$) are two real numbers with $f(a)f(b) \neq 0$ and $f(x), f'(x), ..., f^{(n)}(x)$ are the successive derivatives of $f(x)$. Let $\Delta N = N(a) - N(b)$, where $N(a)$ and $N(b)$ are the numbers of sign-changes of $f(a), f'(a), ..., f^{(n)}(a)$ and $f(b), f'(b), ..., f^{(n)}(b)$, respectively. Then, the number of real roots of the equation $f(x) = 0$ in (a, b) is ΔN, or less than ΔN by a positive even number. Particularly, we have*

(1) if $\Delta N = 0$, there are no real roots of $f(x) = 0$ in (a, b);

(2) if $\Delta N = 1$, there is exactly one real root of $f(x) = 0$ in (a, b).

Let a polynomial $f(x)$ and a small enough interval (a, b) be given, where a, b are rational numbers and suppose that σ is in (a, b). In order to prove $f(\sigma) \neq 0$, we prove $f(x)$ has no real roots in (a, b). In order to prove $f(\sigma) > 0$, we first prove $f(x)$ has no real roots in (a, b) and then choose a rational number $x_0 \in (a, b)$ and check $f(x_0) > 0$. For the reason why the Budan-Fourier theorem is more efficient than the Sturm theorem on large polynomials, we refer to [3, 7].

3 Analysis on the real solutions of the system (3)

To solve the system (3), we solve the equations first by pseudo-division, and then verify the inequation and the inequalities by the Budan-Fourier theorem. First of all, we have

$$L(1) = b_5 + 4a_3 b_1.$$

Substituting $b_5 = -4a_3 b_1$ into $L(2)$ gives

$$L(2) = a_3 b_1 (2a_9 - 3b_6 - 4b_4 + 10a_3^2 - 4a_1 a_3 - 18a_7).$$

For a fine focus of order greater than 2, we must have $L(2) = 0$, and we have three options. If $b_1 = 0$, the origin is a center, as pointed out by James and Lloyd in

[6]. If we take $2a_9 - 3b_6 - 4b_4 + 10a_3^2 - 4a_1a_3 - 18a_7 = 0$, the computation followed will be very heavy. In fact, we have tried this option without obtaining any results and we wonder whether or not this case can be solved by existing computer algebra systems and methods. So, we take $a_3 = 0$, the option that all researchers chose for this system [6, 10, 9], which gives

$$L(3) = -a_1b_1(a_7 + b_4)(2a_9 + 7b_4 - 9a_7).$$

The conditions that $a_1 \neq 0$ and $b_1 \neq 0$ must be satisfied, otherwise the origin will be a center. Obviously the system (3) is now transformed into the following two systems:

$$\begin{cases} b_5 = -4a_3b_1, \\ a_3 = 0, \\ a_7 + b_4 = 0, \\ L(4) = 0, ..., L(7) = 0, \\ L(8) \neq 0, \\ b_8 > 0,\ b_8^2 - a_8^2 > 0, \end{cases} \tag{5}$$

and

$$\begin{cases} b_5 = -4a_3b_1, \\ a_3 = 0, \\ 2a_9 + 7b_4 - 9a_7 = 0, \\ L(4) = 0, ..., L(7) = 0, \\ L(8) \neq 0, \\ b_8 > 0,\ b_8^2 - a_8^2 > 0. \end{cases} \tag{6}$$

3.1 Analysis on the system (5)

In this subsection, we discuss the real solutions to the system (5). James and Lloyd [6] gave one set of *sufficient* conditions for the system (5) to have real solutions. Ning, Ma, Kwek and Zheng [10] gave two sets of *sufficient* conditions for the origin to be a fine focus of order 8 of the system (1), which improved the result by James and Lloyd.

In this subsection, we shall give the *necessary and sufficient* conditions for the system (5) to have real solutions and find that $b_4 < 0$ is a neglected condition in the literature.

We solve the equations in the system (5) one by one eliminating one variable each time. The ordering on variables that we take is $a_7 \prec a_8 \prec b_8 \prec b_6 \prec a_9 \prec b_4$. Note that different orderings may only cause different computational complexity but the results under different orderings should be equivalent to one another.

First, letting $a_7 = -b_4$ and substituting it into the other equations, we have

$$L(4) = a_1b_1a_9(b_6C_1 - 20a_8C_2) = 0,$$

where $C_1 = 13a_9 + 60b_4,\ C_2 = a_9 + 4b_4$. Then, let $a_8 = b_6C_1/(20C_2)$ and the process of elimination followed is

$$\begin{aligned} &L(5) = a_1b_1a_9b_6(C_3 + 48b_8C_2C_4)/(10C_2) = 0,\ b_8 = -C_3/(48C_2C_4); \\ &L(6) = a_1b_1a_9b_6(25C_5 - 36b_6^2C_4^2C_6)/(25C_2C_4^2) = 0,\ b_6^2 = 25C_5/(36C_4^2C_6); \\ &L(7) = a_1b_1a_9b_6C_7/(12C_2C_4^3C_6) = 0; \\ &L(8) = a_1b_1a_9b_6C_8/(6C_2^2C_4^4C_6^2) \neq 0, \end{aligned}$$

where

$$\begin{aligned}
C_3 &= 601a_9^3 + 7240b_4a_9^2 + 30480b_4^2a_9 + 43200b_4^3,\\
C_4 &= 2a_9 + 15b_4,\\
C_5 &= 77851a_9^6 + 2086817a_9^5b_4 + 24208900a_9^4b_4^2 + 155084544a_9^3b_4^3 + 568011840a_9^2b_4^4\\
&\quad + 1104076800a_9b_4^5 + 870912000b_4^6,\\
C_6 &= 149a_9^2 + 1960a_9b_4 + 5600b_4^2,\\
C_7 &= 1324524586a_9^{10} - 5941780227b_4a_9^9 + 447644512436b_4^2a_9^8 + 33468743564464b_4^3a_9^7\\
&\quad + 642634655826240b_4^4a_9^6 + 6325502424166400b_4^5a_9^5 + 37257726560256000b_4^6a_9^4\\
&\quad + 136983308014080000b_4^7a_9^3 + 308644781875200000b_4^8a_9^2\\
&\quad + 389728972800000000b_4^9a_9 + 210691031040000000b_4^{10},
\end{aligned}$$

$$\begin{aligned}
C_8 &= 5551042374556469a_9^{15} + 155659908382453331288064000 0a_9^5b_4^{10}\\
&\quad + 443096034533934760157184000a_9^6b_4^9 + 93550894918369810328064000a_9^7b_4^8\\
&\quad + 14602339377424355028464640a_9^8b_4^7 + 1654282868123736059396096a_9^9b_4^6\\
&\quad + 130705741923117748610112a_9^{10}b_4^5 + 6667217622879185218704a_9^{11}b_4^4\\
&\quad + 189179045049358907992a_9^{12}b_4^3 + 2722007338393061076a_9^{13}b_4^2\\
&\quad + 11771952455795238 3a_9^{14}b_4 + 691671456417872609280000000 0a_9b_4^{14}\\
&\quad + 918181810868644085760000000 0a_9^2b_4^{13} + 738254619338063236300800000 0a_9^3b_4^{12}\\
&\quad + 401252638670271305809920000 0a_9^4b_4^{11} + 238191840255069388800000000 0b_4^{15}.
\end{aligned}$$

By letting $a_9 = \sigma b_4$, C_7 is transformed into a polynomial in σ with degree 10, $C_7 = b_4^{10}\sum_{i=0}^{10} A_i\sigma^i$. Suppose $b_4 \neq 0$ ($b_4 = 0$ implies $a_9 = 0$ and $L(8) = 0$), we find that the polynomial has four real zeros $\sigma_1 < \sigma_2 < \sigma_3 < \sigma_4$. By the Budan-Fourier theorem (or Sturm's theorem), we locate the zeros and check the following inequalities and inequations at the four zeros:

$$C_2 \neq 0,\ C_4 \neq 0,\ C_6 \neq 0,\ C_8 \neq 0\ (L(8) \neq 0),\ b_8^2 - a_8^2 > 0.$$

The above relations all hold at the four zeros. To make $b_6^2 > 0$ true, C_5C_6 must be positive. And we find that $C_5(\sigma_i)C_6(\sigma_i) > 0$ only for $i = 1$ and $i = 4$, where

$$\sigma_1 \in \left(\frac{-17166571}{2500000}, \frac{-1716657}{250000}\right) \text{ and } \sigma_4 \in \left(\frac{-11335691}{5000000}, \frac{-11335689}{5000000}\right).$$

Finally, we check the sign of b_8 at σ_1 and σ_4. Because $C_2(\sigma_i)C_3(\sigma_i)C_4(\sigma_i) > 0$ for $i = 1, 4$ and $\operatorname{sign}(b_8(\sigma_i)) = \operatorname{sign}(-b_4C_2(\sigma_i)C_3(\sigma_i)C_4(\sigma_i))$, to make $b_8 > 0$ true, $b_4 < 0$ must hold. Thus, we have

Theorem 1. *The system* (5) *has real solutions if and only if* $b_4 < 0$, $a_9 = \sigma_1 b_4$ (*or* $a_9 = \sigma_4 b_4$) *and the following equalities hold*

$$b_5 = -4a_3b_1, a_3 = 0, a_7 = -b_4, a_8 = \frac{b_6C_1}{20C_2}, b_8 = \frac{-C_3}{48C_2C_4}, b_6^2 = \frac{25C_5}{36C_4^2C_6}. \tag{7}$$

Corollary 1. *For* $b_4 < 0$ *and* $a_9 = \sigma_1 b_4$ (*or* $a_9 = \sigma_4 b_4$), *if the conditions* (7) *are satisfied, the origin is a fine focus of order eight of the system* (1).

By a simple discussion on the sequential perturbations on the variables, we have

Theorem 2. *For the system* (1)*, if* $b_4 < 0$*,* $a_9 = \sigma_1 b_4$ *(or* $a_9 = \sigma_4 b_4$*) and the conditions* (7) *are satisfied, eight limit cycles can be bifurcated from the origin by perturbing* $a_9, b_6, b_8, a_8, a_7, a_3$ *and* b_5 *sequentially.*

Note that $b_4 < 0$ is a neglected condition in the literature.

3.2 Analysis on the system (6)

In this subsection, we discuss the real solutions to the system (6). Ma and Ning [9] gave 10 sets of *sufficient* conditions for the system (6) to have real solutions. We shall obtain the *necessary and sufficient* conditions for the system (6) to have real solutions and find that the result by Ma and Ning is strictly sufficient but not necessary.

To simplify the descriptions, we use the following notations. If $L(i)$ is a polynomial with k terms, it is denoted by $L(i) = (k)$; and if $L(i)$ is the product of m polynomials, each of which has k_i $(1 \le i \le m)$ terms, it is denoted by $L(i) = (k_1) \cdots (k_m)$. If two different polynomials both have k terms, they may be denoted by (k) and $(k)^*$, respectively, to indicate the difference.

Analogously, we first solve the equations in the system (6) by eliminating variables one by one. The ordering we take is $a_7 \prec a_9 \prec a_8 \prec b_8 \prec b_4 \prec b_6$. Substituting $a_7 = \frac{1}{9}(2a_9 + 7b_4)$ into the equations in the system (6), we have the following table (where $\mathrm{d}(x_i)$ means the degree of a polynomial with respect to x_i).

	$\mathrm{d}(a_9)$	$\mathrm{d}(a_8)$	$\mathrm{d}(b_8)$	$\mathrm{d}(b_4)$	$\mathrm{d}(b_6)$
$L(4) = (13)$	3	1	1	3	1
$L(5) = (35)$	4	2	2	4	2
$L(6) = (75)$	5	3	3	5	3
$L(7) = (140)$	6	4	4	6	4
$L(8) = (238)$	7	5	5	7	5

To eliminate a_9, $L(i)$ $(5 \le i \le 8)$ is pseudo-divided by $L(4)$ with respect to a_9. We denote the pseudo-remainders still by $L(i)$ $(5 \le i \le 8)$, respectively. Then, $L(i)$ $(i = 4, 6, 7, 8)$ is pseudo-divided by $L(5)$ with respect to a_9. Continuing this process until the degree in a_9 reaches 0, we get that

$$L(4) = b_4(106),\ L(5) = b_4(b_4 + b_6)(10)^2(104),\ L(6) = b_4(b_4 + b_6)(10)(104)^*,$$

$$L(7) = b_4(b_4 + b_6)(10)(148),\ L(8) = b_4(b_4 + b_6)(10)(202).$$

	$\mathrm{d}(a_9)$	$\mathrm{d}(a_8)$	$\mathrm{d}(b_8)$	$\mathrm{d}(b_4)$	$\mathrm{d}(b_6)$
(106)	1	4	4	6	5
(10)	0	2	2	2	2
(104)	0	5	6	6	7
$(104)^*$	0	5	6	6	7
(148)	0	6	7	7	8
(202)	0	7	8	8	9

Denote the polynomial with 106 terms by G_1 which can be written in the form of

$$G_1 = \sum_{i=0}^{4} f_i(a_9, b_4, b_6, b_8)a_8^i,$$

where degree$(G_1, a_9) = 1$. In the process of performing pseudo-division to obtain G_1, we get some *initials* (leading coefficients) [13, 14] at each step. We denote the product of those initials by I_1.

Now, we take the similar process to eliminate a_8 using $L(5), L(6)$ and $L(7)$. The computations give that

$$L(5) = b_4(68)^2(965),\ L(6) = b_6(199)(77)^2(211)^2,$$

$$L(7) = b_6(340)(77)(211),\ L(8) = b_6(367)(77)(211).$$

	d(a_9)	d(a_8)	d(b_8)	d(b_4)	d(b_6)
(68)	0	0	8	11	11
(965)	0	1	26	30	30
(77)	0	0	11	11	10
(211)	0	0	16	20	19
(199)	0	0	15	19	19
(340)	0	0	21	25	25
(367)	0	0	22	26	26

Denote the polynomial with 965 terms by G_2 which can be written in the form of

$$G_2 = \sum_{i=0}^{26} g_i(b_4, b_6, a_8)b_8^i.$$

Analogously, we denote by I_2 the product of those initials occurring in the process of performing pseudo-division to obtain G_2.

The computations for eliminating b_8 give that

$$L(6) = (328)^2(817),\ L(7) = (75)(137)(121)^2(367),\ L(8) = (178)(137)(121)(367).$$

	d(a_9)	d(a_8)	d(b_8)	d(b_4)	d(b_6)
(328)	0	0	0	327	327
(817)	0	0	1	408	408
(75)	0	0	0	74	74
(178)	0	0	0	177	177

Denote the polynomial with 817 terms, 75 terms and 178 terms by G_3, G_4 and G_5, respectively, which can be written respectively in the form of

$$G_3 = \sum_{i=0}^{408} h_i(b_8)b_4^i b_6^{408-i},\ G_4 = \sum_{i=0}^{74} \alpha_i b_4^i b_6^{74-i},\ G_5 = \sum_{i=0}^{177} \beta_i b_4^i b_6^{177-i},$$

where α and β are large integers. Analogously, we denote by I_3 the product of those initials occurring in the process of performing pseudo-division to obtain G_3. Because the representations of $G_1, ..., G_5$ on computer cost 2.5 Mega bytes of memory, we do not give the detailed information here.

It is easy to see that the solutions of the system (6) are the same as those of the following system

$$G_1 = G_2 = G_3 = G_4 = 0, G_5 \neq 0, b_8 > 0, b_8^2 - a_8^2 > 0, J = I_1 I_2 I_3 \neq 0.$$

By letting $b_6 = \tau b_4$, we have

$$G_4 = b_4^{74} \sum_{i=0}^{74} \alpha_i \tau^{74-i} = 0.$$

Suppose $b_4 \neq 0$, the polynomial has 16 distinct real zeros which are

$$\begin{aligned}
&\tau_1 \in \left(-\frac{492238219}{5000000}, -\frac{492238217}{5000000}\right), && \tau_2 \in \left(-\frac{47502533}{1250000}, -\frac{19001013}{500000}\right),\\
&\tau_3 \in \left(-\frac{110420651}{5000000}, -\frac{110420649}{5000000}\right), && \tau_4 \in \left(-\frac{17336617}{1250000}, -\frac{34673233}{2500000}\right),\\
&\tau_5 \in \left(-\frac{14207291}{1250000}, -\frac{28414581}{2500000}\right), && \tau_6 \in \left(-\frac{27971559}{2500000}, -\frac{13985779}{1250000}\right),\\
&\tau_7 \in \left(-\frac{2036071}{200000}, -\frac{50901773}{5000000}\right), && \tau_8 \in \left(-\frac{22842879}{2500000}, -\frac{11421439}{1250000}\right),\\
&\tau_9 \in \left(-\frac{19599193}{2500000}, -\frac{2449899}{312500}\right), && \tau_{10} \in \left(-\frac{3517249}{500000}, -\frac{4396561}{625000}\right),\\
&\tau_{11} \in \left(-\frac{33599479}{5000000}, -\frac{33599477}{5000000}\right), && \tau_{12} \in \left(-\frac{5947783}{1000000}, -\frac{29738913}{5000000}\right),\\
&\tau_{13} \in \left(-\frac{12642573}{2500000}, -\frac{3160643}{625000}\right), && \tau_{14} \in \left(-\frac{22741473}{5000000}, -\frac{22741471}{5000000}\right),\\
&\tau_{15} \in \left(-\frac{35837}{312500}, -\frac{57339}{500000}\right), && \tau_{16} \in \left(\frac{1079877}{2500000}, \frac{539939}{1250000}\right).
\end{aligned}$$

First, we check $G_5 \neq 0$ ($L(8) \neq 0$) and $J = I_1 I_2 I_3 \neq 0$ at the 16 zeros by the Budan-Fourier theorem and find that the two inequations do hold at all those real zeros.

Second, we check $b_8 > 0$ and $b_8^2 - a_8^2 > 0$ at the zeros and find that the two inequalities both hold at 11 of those zeros which are $\tau_2, \tau_3, \tau_4, \tau_7, \tau_8, \tau_9, \tau_{10}, \tau_{11}, \tau_{14}, \tau_{15}$ and τ_{16}. Because

$$G_3 = l_1(\tau_i) b_8 + b_4 l_0(\tau_i) = 0, \quad (i = 1, ..., 16)$$

where l_1 and l_0 are the leading coefficient and trailing coefficient of G_3 with respect to b_8, respectively, by checking the signs of l_1 and l_0 at the zeros, we obtain that if $\tau \in \{\tau_2, \tau_3, \tau_4, \tau_7, \tau_8, \tau_9\}$ (or $\{\tau_{10}, \tau_{11}, \tau_{14}, \tau_{15}, \tau_{16}\}$), $b_4 > 0$ (or $b_4 < 0$) must hold for b_8 to be positive. Thus, we obtain

Theorem 3. *The system* (6) *has real solutions if and only if* (I) $b_4 > 0$, $b_6 = \tau b_4$ ($\tau \in \{\tau_2, \tau_3, \tau_4, \tau_7, \tau_8, \tau_9\}$) *and the conditions* (8) *are satisfied; or* (II) $b_4 < 0$, $b_6 = \tau b_4$ ($\tau \in \{\tau_{10}, \tau_{11}, \tau_{14}, \tau_{15}, \tau_{16}\}$) *and the conditions* (8) *are satisfied.*

$$b_5 = -4a_3 b_1, \; a_3 = 0, \; a_7 = \frac{1}{9}(2a_9 + 7b_4), \; G_1 = 0, \; G_2 = 0, \; G_3 = 0. \tag{8}$$

Corollary 2. *For* $b_4 > 0$ *and* $b_6 = \tau b_4$ ($\tau \in \{\tau_2, \tau_3, \tau_4, \tau_7, \tau_8, \tau_9\}$) *or* $b_4 < 0$ *and* $b_6 = \tau b_4$ ($\tau \in \{\tau_{10}, \tau_{11}, \tau_{14}, \tau_{15}, \tau_{16}\}$), *if the conditions* (8) *are satisfied, the origin is a fine focus of order eight of the system* (1).

Perturbing on the variables sequentially, we have

Theorem 4. *For the system* (1)*, if* $b_4 > 0$ *and* $b_6 = \tau b_4$ $(\tau \in \{\tau_2, \tau_3, \tau_4, \tau_7, \tau_8, \tau_9\})$ *or* $b_4 < 0$ *and* $b_6 = \tau b_4$ $(\tau \in \{\tau_{10}, \tau_{11}, \tau_{14}, \tau_{15}, \tau_{16}\})$*, and the conditions* (8) *are satisfied, then eight limit cycles can be bifurcated from the origin by perturbing* $b_6, b_8, a_8, a_9, a_7, a_3$ *and* b_5 *sequentially.*

4 Main results

Under the limitation that $a_3 = 0$, the system (3) is transformed equivalently to the systems (5) and (6). By the results in Section 3, we have the following results.

Theorem 5. *Suppose* $a_3 = 0$.

(I)*. The system* (3) *has real solutions if and only if one of the following sets of conditions hold.*

1. $\{b_4 < 0,\ a_9 = \sigma_1 b_4\ (or\ a_9 = \sigma_4 b_4),\ b_5 = -4a_3 b_1,\ a_3 = 0,\ a_7 = -b_4,\ a_8 = \dfrac{b_6 C_1}{20 C_2},$ $b_8 = \dfrac{-C_3}{48 C_2 C_4},\ b_6^2 = \dfrac{25 C_5}{36 C_4^2 C_6}\},$

2. $\{b_4 > 0,\ b_6 = \tau_i b_4\ (i = 2, 3, 4, 7, 8, 9),\ b_5 = -4a_3 b_1,\ a_3 = 0,\ a_7 = \dfrac{1}{9}(2a_9 + 7b_4),$ $G_1 = 0,\ G_2 = 0,\ G_3 = 0\}$, *and*

3. $\{b_4 < 0,\ b_6 = \tau_i b_4\ (i = 10, 11, 14, 15, 16),\ b_5 = -4a_3 b_1,\ a_3 = 0,\ a_7 = \dfrac{1}{9}(2a_9 + 7b_4),$ $G_1 = 0,\ G_2 = 0,\ G_3 = 0\}.$

(II)*. The above conditions are the necessary and sufficient conditions for the origin to be an eighth-order fine focus of the system* (1).

(III)*. The above conditions are also the necessary and sufficient conditions for the system* (1) *to have eight small-amplitude limit cycles.*

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