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An Algorithm for Isolating the Real Solutions of Semi-algebraic Systems

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We propose an algorithm for isolating the real solutions of semi-algebraic systems, which has been implemented as a Maple-program `realzero`. The performance of `realzero` in solving some examples from various applications is presented and the timings are reported.

1. Introduction

By semi-algebraic systems, we mean systems of polynomial equations, inequalities and inequations. More precisely, we call

$$\begin{cases} p_1(x_1, x_2, \ldots, x_s) = 0, \ldots, p_n(x_1, x_2, \ldots, x_s) = 0, \\ g_1(x_1, x_2, \ldots, x_s) \ge 0, \ldots, g_r(x_1, x_2, \ldots, x_s) \ge 0, \\ g_{r+1}(x_1, x_2, \ldots, x_s) > 0, \ldots, g_t(x_1, x_2, \ldots, x_s) > 0, \\ h_1(x_1, x_2, \ldots, x_s) \ne 0, \ldots, h_m(x_1, x_2, \ldots, x_s) \ne 0, \end{cases} \tag{1}$$

a *semi-algebraic system* (SAS for short), where $n, s \ge 1, r, t, m \ge 0$ and p_i, g_j, h_k are all polynomials in $x_1, \ldots, x_s$ with integer coefficients. Furthermore, we always assume that $\{p_1, \ldots, p_n\}$ has only a finite number of common zeros.

Many problems in both practice and theory can be reduced to problems of solving SAS. For example, we may mention some special cases of the “p-3-p” problem (Folke, 1994) which originates from computer vision, the problem of constructing limit cycles for plane differential systems (Ma and Zheng, 1994) and the problem of automated discovering and proving for geometric inequalities (Yang *et al.*, 1999, 2001). Moreover, many problems in geometry, topology and differential dynamical systems are expected to be solved by translating them into certain semi-algebraic systems. There are two classical methods, Tarski’s method (Tarski, 1951) and the cylindrical algebraic decomposition method proposed by Collins (1975), for solving semi-algebraic systems.

Counting and isolating real solutions are two key problems in the study of the real solutions of a SAS from the viewpoint of symbolic computation. Some effective methods for attacking the first problem are those using trace forms or the rational univariate representation (Pedersen *et al.*, 1993; Gonzalez-Vega *et al.*, 1999) and the algorithm

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proposed by Xia and Hou (2002). Usually, these methods may suggest some algorithms for attacking the second problem. In this paper, combining the algorithms such as the Ritt–Wu method (Wu, 1986) and the WR algorithm (Yang *et al.*, 1995) for solving systems of polynomial equations with the Uspensky algorithm (Collins and Loos, 1983) for isolating real zeros of a univariate polynomial, we present an algorithm for isolating the real solutions of semi-algebraic systems which, in some sense, can be viewed as a generalization of the Uspensky algorithm. Our algorithm appears to be practical in solving many problems from various applications though it is not complete in theory.

2. Basic Algorithm

In this paper, all the polynomials, if not specified, are in $\mathbb{Z}[x_1, \ldots, x_s]$. For any polynomial P with positive degree, the *leading variable* x_l of P is the one with greatest index l that effectively appears in P. By a *triangular set*, we mean a set of polynomials $\{f_i(x_1, \ldots, x_i), f_{i+1}(x_1, \ldots, x_{i+1}), \ldots, f_l(x_1, \ldots, x_l)\}$ in which the leading variable of f_j is x_j.

If the ideal generated by $p_1, \ldots, p_n$ is zero dimensional, then it is well known that the Ritt–Wu method, Gröbner basis methods or subresultant methods can be used to transform the system of equations into one or more systems in triangular form (see, for example, Buchberger, 1985; Wu, 1986; Yang *et al.*, 1995; Wang, 1998; Aubry *et al.*, 1999). Therefore, in Sections 2 and 3, we only consider triangular sets and the problem we discuss is to isolate the real solutions of the following system

$$\begin{cases} f_1(x_1) = 0, \\ f_2(x_1, x_2) = 0, \\ \cdots\cdots \\ f_s(x_1, x_2, \ldots, x_s) = 0, \\ g_1(x_1, x_2, \ldots, x_s) \geq 0, \ldots, g_r(x_1, x_2, \ldots, x_s) \geq 0, \\ g_{r+1}(x_1, x_2, \ldots, x_s) > 0, \ldots, g_t(x_1, x_2, \ldots, x_s) > 0, \\ h_1(x_1, x_2, \ldots, x_s) \neq 0, \ldots, h_m(x_1, x_2, \ldots, x_s) \neq 0, \end{cases} \tag{2}$$

where $s \geq 1, r, t, m \geq 0$ and $\{f_1, f_2, \ldots, f_s\}$ is a normal ascending chain (Yang *et al.*, 1995) (also see Definition 2.3 and Remark 4 in this section). We call a system in this form a *triangular semi-algebraic system* (TSA for short).

Definition 2.1. Given a polynomial $g(x)$, let $\text{resultant}(g, g'_x, x)$ be the Sylvester resultant of g and g'_x with respect to x, where g'_x means the derivative of $g(x)$ with respect to x. We call it the *discriminant* of g with respect to x and denote it by $\text{dis}(g, x)$ or simply by $\text{dis}(g)$ if its meaning is clear.

Definition 2.2. Given a polynomial g and a triangular set $\{f_1, f_2, \ldots, f_s\}$, let

$$r_s := g, \quad r_{s-i} := \text{resultant}(r_{s-i+1}, f_{s-i+1}, x_{s-i+1}), \quad i = 1, 2, \ldots, s;$$

$$q_s := g, \quad q_{s-i} := \text{prem}(q_{s-i+1}, f_{s-i+1}, x_{s-i+1}), \quad i = 1, 2, \ldots, s,$$

where $\text{resultant}(p, q, x)$ means the Sylvester resultant of p, q with respect to x and $\text{prem}(p, q, x)$ means the pseudo-remainder of p divided by q with respect to x.

Let r_{i-1} and $q_{i-1} (1 \leq i \leq s)$ be denoted by $\text{res}(g, f_s, \ldots, f_i)$ and $\text{prem}(g, f_s, \ldots, f_i)$ and called the *resultant* and *pseudo-remainder* of g with respect to the triangular set $\{f_i, f_{i+1}, \ldots, f_s\}$, respectively.

DEFINITION 2.3. Given a triangular set $\{f_1, f_2, \ldots, f_s\}$, denote by $I_i (i = 1, \ldots, s)$ the leading coefficient of f_i in x_i. A triangular set $\{f_1, f_2, \ldots, f_s\}$ is called a *normal ascending chain* if $\mathrm{res}(I_i, f_{i-1}, \ldots, f_1) \neq 0$ for $i = 2, \ldots, s$. Note that $I_1 \neq 0$ follows from the definition of a triangular set.

REMARK 1. A normal ascending chain is also called a *regular chain* by Kalkbrener (1993) and a *regular set* by D. M. Wang (2000).

DEFINITION 2.4. Let a TSA be given as defined in (2), called T. For every $f_i (i \geq 2)$, let $\mathrm{CP}_{f_2} = \mathrm{dis}(f_2, x_2)$ and

$$\mathrm{CP}_{f_i} = \mathrm{res}(\mathrm{dis}(f_i, x_i), f_{i-1}, f_{i-2}, \ldots, f_2), \qquad i > 2.$$

For any $q \in \{g_j \mid 1 \leq j \leq t\} \bigcup \{h_k \mid 1 \leq k \leq m\}$, let

$$\mathrm{CP}_q = \begin{cases} \mathrm{res}(q, f_s, f_{s-1}, \ldots, f_2), & \text{if } s > 1, \\ q, & \text{if } s = 1. \end{cases}$$

We define $\mathrm{CP}_T(x_1) = \prod_{2 \leq i \leq s} \mathrm{CP}_{f_i} \cdot \prod_{1 \leq j \leq t} \mathrm{CP}_{g_j} \cdot \prod_{1 \leq k \leq m} \mathrm{CP}_{h_k}$, and call it the *critical polynomial* of the system T with respect to x_1. We also denote $\mathrm{CP}_T(x_1)$ by CP or $\mathrm{CP}(x_1)$ if its meaning is clear.

REMARK 2. Let a TSA T be given and denote by T_1 the system formed by deleting $f_1(x_1)$ from T. In T_1, we view x_1 as a parameter and let it vary continuously on the real number axis. From Theorem 2.1, we know that the number of distinct real solutions of T_1 will remain fixed provided that x_1 varies on an interval in which there are no real zeros of $\mathrm{CP}_T(x_1)$. That is why $\mathrm{CP}_T(x_1)$ is called the *critical polynomial* of the system T.

DEFINITION 2.5. A TSA is *regular* if $\mathrm{resultant}(f_1(x_1), \mathrm{CP}(x_1), x_1) \neq 0$.

REMARK 3. According to Definition 2.5, for a regular TSA, no $\mathrm{CP}_{h_k} (1 \leq k \leq m)$ has common zeros with $f_1(x_1)$, which implies that every solution of $\{f_1 = 0, \ldots, f_s = 0\}$ satisfies $h_k \neq 0 (1 \leq k \leq m)$. Thus if a TSA is regular we can omit the h_k's in it without loss of generality. Similarly, every solution of $\{f_1 = 0, \ldots, f_s = 0\}$ satisfies $g_j \neq 0 (1 \leq j \leq t)$. That is to say, each of the inequalities $g_j \geq 0 (1 \leq j \leq r)$ in a regular TSA can be treated as $g_j > 0$.

2.1. REGULAR TSAs

Given two polynomials $p(x), q(x) \in \mathbb{Z}[x]$, suppose $p(x)$ and $q(x)$ have no common zeros, i.e. $\mathrm{resultant}(p, q, x) \neq 0$, and $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ are all distinct real zeros of $p(x)$. By the modified Uspensky algorithm (Collins and Loos, 1983), we can obtain a sequence of intervals, $[a_1, b_1], \ldots, [a_n, b_n]$, satisfying

(1) $\alpha_i \in [a_i, b_i]$ for $i = 1, \ldots, n$,
(2) $[a_i, b_i] \bigcap [a_j, b_j] = \emptyset$ for $i \neq j$,
(3) $a_i, b_i (1 \leq i \leq n)$ are all rational numbers, and
(4) the maximal size of each isolating interval can be less than any positive number given in advance. Because $p(x)$ and $q(x)$ have no common zeros, the intervals can also satisfy
(5) no zeros of $q(x)$ are in any $[a_i, b_i]$.

In the following we denote an algorithm to do this by $\texttt{nearzero}(p, q, x)$, or $\texttt{nearzero}(p, q, x, \epsilon)$ if the maximal size of the isolating intervals is specified to be not greater than a positive number ϵ.

THEOREM 2.1. *Let a regular* TSA *be given. Suppose $f_1(x_1)$ has n distinct real zeros; then, by calling $\texttt{nearzero}(f_1, \mathrm{CP}(x_1), x_1)$ we can obtain a sequence of intervals, $[a_1, b_1], \ldots, [a_n, b_n]$, satisfying, for any $[a_i, b_i](1 \leq i \leq n)$ and any $\beta, \gamma \in [a_i, b_i]$,*

(1) *if $s > 1$, then the system*

$$\begin{cases} f_2(\beta, x_2) = 0, \ldots, f_s(\beta, x_2, \ldots, x_s) = 0, \\ g_1(\beta, x_2, \ldots, x_s) > 0, \ldots, g_t(\beta, x_2, \ldots, x_s) > 0, \end{cases}$$

and the system

$$\begin{cases} f_2(\gamma, x_2) = 0, \ldots, f_s(\gamma, x_2, \ldots, x_s) = 0, \\ g_1(\gamma, x_2, \ldots, x_s) > 0, \ldots, g_t(\gamma, x_2, \ldots, x_s) > 0, \end{cases}$$

have the same number of distinct real solutions and,

(2) *if $s = 1$, then for any $g_j(1 \leq j \leq t)$, $\mathrm{sign}(g_j(\beta)) = \mathrm{sign}(g_j(\gamma))$, where $\mathrm{sign}(x)$ is 1 if $x > 0$, -1 if $x < 0$ and 0 if $x = 0$.*

PROOF. Because the TSA is regular, f_1 has no common zeros with the critical polynomial $\mathrm{CP}(x_1)$. So, by calling $\texttt{nearzero}(f_1, \mathrm{CP}(x_1), x_1)$ we can get a sequence of intervals which satisfies the five conditions of $\texttt{nearzero}$. If $s = 1$, the conclusion is obvious. So, suppose $s > 1$. Because $\mathrm{CP}(x_1) = \prod_{2 \leq i \leq s} \mathrm{CP}_{f_i} \cdot \prod_{1 \leq j \leq t} \mathrm{CP}_{g_j}$ has no zeros on $[a_i, b_i]$, clearly, the sign of each CP_{f_i} and CP_{g_j} is invariant on the interval $[a_i, b_i]$.

First of all, $\mathrm{CP}_{f_2} = \mathrm{dis}(f_2, x_2) \neq 0$ on the interval $[a_i, b_i]$ implies that the number of distinct real zeros of f_2 is invariant on $[a_i, b_i]$. Furthermore, $\mathrm{CP}_{f_3} \neq 0$ on $[a_i, b_i]$ implies that if $f_2 = 0$, then $\mathrm{dis}(f_3, x_3) \neq 0$ on $[a_i, b_i]$, which means that the number of distinct real solutions of equations $\{f_2 = 0, f_3 = 0\}$ is invariant on $[a_i, b_i]$. Continuing in this way, we see that the number of distinct real solutions of equations $\{f_2 = 0, \ldots, f_s = 0\}$ is invariant on $[a_i, b_i]$. Secondly, $\mathrm{CP}_{g_j} \neq 0$ on $[a_i, b_i]$ implies that if $\{f_2 = 0, \ldots, f_s = 0\}$, then $g_j \neq 0$ on $[a_i, b_i]$, which means that the number of distinct real solutions of the given TSA without f_1 is invariant on $[a_i, b_i]$. The proof is complete. □

2.2. IRREGULAR TSAs

In this subsection, we discuss irregular TSAs and give a theorem which guarantees that we can always assume a given system to be regular, without loss of generality. Our main tool is the WR algorithm (Yang *et al.*, 1995). Here are some related definitions and results.

DEFINITION 2.6. (YANG *ET AL.*, 1995) A normal ascending chain $\{f_1, \ldots, f_s\}$ is *simplicial* with respect to a polynomial g if either $\mathrm{prem}(g, f_s, \ldots, f_1) = 0$ or $\mathrm{res}(g, f_s, \ldots, f_1) \neq 0$.

THEOREM 2.2. (YANG *ET AL.*, 1995) *For a triangular set $AS : \{f_1, \ldots, f_s\}$ and a polynomial g, there is an algorithm which can decompose AS into some normal ascending chains $AS_i : \{f_{i1}, f_{i2}, \ldots, f_{is}\}(1 \leq i \leq n)$, such that every chain is simplicial with respect to g and this decomposition satisfies that $Zero(AS) = \bigcup_{1 \leq i \leq n} Zero(AS_i)$, where $Zero(\cdot)$ means the set of zeros of a given system.*

REMARK 4. This decomposition is called the WR decomposition of AS with respect to g and the algorithm is called the WR algorithm. D. M. Wang (2000) proposed a similar decomposition algorithm. By Theorem 2.2, we always consider the triangular set $\{f_1, f_2, \ldots, f_s\}$ that appears in a TSA as a normal ascending chain, without loss of generality.

DEFINITION 2.7. Given a polynomial with general symbolic coefficients, $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$, the following $2n \times 2n$ matrix in terms of the coefficients,

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_n & & & & \\ 0 & na_0 & (n-1)a_1 & \cdots & a_{n-1} & & & & \\ & a_0 & a_1 & \cdots & a_{n-1} & a_n & & & \\ & 0 & na_0 & \cdots & 2a_{n-2} & a_{n-1} & & & \\ & & & \cdots & \cdots & & & & \\ & & & \cdots & \cdots & & & & \\ & & & a_0 & a_1 & a_2 & \cdots & a_n & \\ & & & 0 & na_0 & (n-1)a_1 & \cdots & a_{n-1} & \end{bmatrix}$$

is called the *discrimination matrix* of $f(x)$, and denoted by $\mathrm{Discr}(f)$. Denote by d_k the determinant of the submatrix of $\mathrm{Discr}(f)$, formed by the first k rows and the first k columns for $k = 1, 2, \ldots, 2n$.

DEFINITION 2.8. Let $D_k = d_{2k}, k = 1, \ldots, n$. We call the n-tuple $[D_1, D_2, \ldots, D_n]$ the *discriminant sequence* of $f(x)$. Obviously, the last term D_n is $\mathrm{dis}(f, x)$.

DEFINITION 2.9. (LOOS, 1983) Let D_k^t be the submatrix of $\mathrm{Discr}(f)$, formed by the first $2n-2k$ rows, the first $2n-2k-1$ columns and the $(2n-2k+t)$th column, where $0 \le k \le n-1, 0 \le t \le 2k$. Let $|D_k^t| = \det(D_k^t)$. We call $|D_k^0|$ $(0 \le k \le n-1)$ the kth *principal subresultant* of $f(x)$. Obviously, $|D_k^0| = D_{n-k}(0 \le k \le n-1)$.

DEFINITION 2.10. (LOOS, 1983) Let $Q_{n+1}(f, x) = f(x)$, $Q_n(f, x) = f'(x)$, and for $k = 0, 1, \ldots, n-1$, $Q_k(f, x) = \sum_{t=0}^{k} |D_k^t| x^{k-t} = |D_k^0|x^k + |D_k^1|x^{k-1} + \cdots + |D_k^k|$. We call $\{Q_0(f, x), \ldots, Q_{n+1}(f, x)\}$ the *subresultant polynomial chain* of $f(x)$.

THEOREM 2.3. (YANG *ET AL.*, 1996) *Suppose $\{f_1, f_2, \ldots, f_j\}$ is a normal ascending chain, where K is a field and $f_i \in K[x_1, \ldots, x_i], (i = 1, 2, \ldots, j)$ and $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ is a polynomial in $K[x_1, \ldots, x_i][x]$, let $PD_k = \mathrm{prem}(|D_k^0|, f_j, \ldots, f_1) = \mathrm{prem}(D_{n-k}, f_j, \ldots, f_1), (0 \le k < n)$. If for some $k_0 \ge 0$, $\mathrm{res}(a_0, f_j, \ldots, f_1) \ne 0$ and $PD_0 = \cdots = PD_{k_0-1} = 0, \mathrm{res}(|D_{k_0}^0|, f_j, \ldots, f_1) \ne 0$, then, we have $\gcd(f, f_x') = Q_{k_0}(f, x)$ in $K[x_1, \ldots, x_j]/(f_1, \ldots, f_j)$.*

THEOREM 2.4. *For an irregular TSA T, there is an algorithm which can decompose T into regular systems T_i. Let all the distinct real solutions of a given system be denoted by $Rzero(\cdot)$; then this decomposition satisfies $Rzero(T) = \bigcup Rzero(T_i)$.*

PROOF. For T, $\mathrm{resultant}(f_1, \mathrm{CP}, x_1) = 0$.

- If there is some CP_{h_k} such that $\mathrm{resultant}(f_1, \mathrm{CP}_{h_k}, x_1) = 0$, do the WR decomposition of $\{f_1, \ldots, f_s\}$ with respect to h_k and, without loss of generality, suppose we get two new chains $\{A_1, \ldots, A_s\}$ and $\{B_1, \ldots, B_s\}$, in which $\mathrm{prem}(h_k, A_s, \ldots, A_1) = 0$ but $\mathrm{res}(h_k, B_s, \ldots, B_1) \neq 0$. If we replace $\{f_1, \ldots, f_s\}$ by $\{B_1, \ldots, B_s\}$ in T, the new system is regular and has the same real solutions as those of the original system. Obviously, another system obtained by replacing $\{f_1, \ldots, f_s\}$ with $\{A_1, \ldots, A_s\}$ in T, has no real solutions.

- If there is some CP_{g_j} such that $\mathrm{resultant}(f_1, \mathrm{CP}_{g_j}, x_1) = 0$, do the WR decomposition of $\{f_1, \ldots, f_s\}$ with respect to g_j and suppose we get $\{A_1, \ldots, A_s\}$ and $\{B_1, \ldots, B_s\}$, in which $\mathrm{prem}(g_j, A_s, \ldots, A_1) = 0$ but $\mathrm{res}(g_j, B_s \ldots, B_1) \neq 0$. Now, if $g_j > 0$ in T, we simply replace $\{f_1, \ldots, f_s\}$ by $\{B_1, \ldots, B_s\}$. The new system is regular and has the same real solutions as those of the original system. If $g_j \geq 0$ in T, we first get a new system T_1 by replacing $\{f_1, \ldots, f_s\}$ with $\{B_1, \ldots, B_s\}$ and then, get another new system T_2 by replacing $\{f_1, \ldots, f_s\}$ with $\{A_1, \ldots, A_s\}$ and deleting g_j from it. These two systems are both regular and we have $Rzero(T) = Rzero(T_1) \bigcup Rzero(T_2)$.
- If there is some CP_{f_i} such that $\mathrm{resultant}(f_1, \mathrm{CP}_{f_i}, x_1) = 0$, let $[D_1, \ldots, D_{n_i}]$ be the discriminant sequence of f_i with respect to x_i. First of all, we do the WR decomposition of $\{f_1, \ldots, f_{i-1}\}$ with respect to D_{n_i} and suppose we get $\{A_1, \ldots, A_{i-1}\}$ and $\{B_1, \ldots, B_{i-1}\}$, in which $\mathrm{prem}(f_i, A_{i-1}, \ldots, A_1) = 0$ but $\mathrm{res}(f_i, B_{i-1}, \ldots, B_1) \neq 0$. Step 1, replacing $\{f_1, \ldots, f_{i-1}\}$ with $\{B_1, \ldots, B_{i-1}\}$, we will get a regular system. Step 2, let us consider the system obtained by replacing $\{f_1, \ldots, f_{i-1}\}$ with $\{A_1, \ldots, A_{i-1}\}$ which is still irregular. Consider D_{n_i-1}, the next term in $[D_1, \ldots, D_{n_i}]$. If $\mathrm{res}(D_{n_i-1}, A_{i-1}, \ldots, A_1) = 0$, do the WR decomposition of $\{A_1, \ldots, A_{i-1}\}$ with respect to D_{n_i-1}. Keep repeating the same procedure until at a certain step we have, for certain D_{i_0} and $\{\bar{A}_1, \ldots, \bar{A}_{i-1}\}$, $\mathrm{res}(D_{i_0}, \bar{A}_{i-1}, \ldots, \bar{A}_1) \neq 0$ and $\forall j (i_0 < j \leq n_i), \mathrm{prem}(D_j, \bar{A}_{i-1}, \ldots, \bar{A}_1) = 0$. Note that this procedure must terminate because $\{f_1, \ldots, f_s\}$ being a normal ascending chain implies $\mathrm{res}(I_i, f_{i-1}, \ldots, f_1) \neq 0$ and $D_1 = n_i I_i^2$ implies $\mathrm{res}(D_1, f_{i-1}, \ldots, f_1) \neq 0$. By Theorem 2.3, $\gcd(f_i, f_i') = Q_{n_i - i_0}(f_i, x_i)$ in $K[x_1, \ldots, x_{i-1}]/(\bar{A}_1, \ldots, \bar{A}_{i-1})$. Now, let $\bar{f}_i$ be the pseudo-quotient of f_i divided by $\gcd(f_i, f_i')$ and replace $\{f_1, \ldots, f_{i-1}, f_i\}$ with $\{\bar{A}_1, \ldots, \bar{A}_{i-1}, \bar{f}_i\}$, the new system will be regular. If the new regular systems are $T_j (1 \leq j \leq j_i)$, it is easy to see that $Rzero(T) = \bigcup_{1 \leq j \leq j_i} Rzero(T_j)$.

This completes the proof. □

3. Lifting and Recursion

By Theorem 2.4, we need only to consider regular TSAs. For a regular TSA, by calling `nearzero`$(f_1(x_1), \mathrm{CP}(x_1), x_1)$, we can get a sequence of intervals satisfying the five conditions of the algorithm `nearzero`. How do we make use of these isolating intervals of $f_1(x_1)$ to get those of $f_2(x_2), \ldots, f_s(x_s)$?

Consider $f_2(x_1, x_2)$ and an isolating interval $[a, b]$ of $f_1(x_1)$ obtained by `nearzero` $(f_1(x_1), \mathrm{CP}(x_1), x_1)$ and suppose that $x^{(0)}$ is the zero of $f_1(x_1)$ in $[a, b]$. Let f_2 be viewed as a curve in the plane $\mathbb{R}^2$, by Theorem 2.1, for any $\alpha_1, \alpha_2 \in [a, b]$, the number of intersection points of $x_1 = \alpha_1$ and $f_2(x_1, x_2)$ is equal to that of $x_1 = \alpha_2$ and $f_2(x_1, x_2)$. Especially, that is true for $\alpha_1 = a$ and $\alpha_2 = b$.

Let x_2 be regarded as a function of x_1 implicitly defined by f_2, then

$$\frac{\partial x_2}{\partial x_1} = -\frac{\partial f_2}{\partial x_1} \Big/ \frac{\partial f_2}{\partial x_2}.$$

Noting that

$$\mathrm{res}\left(\frac{\partial f_2}{\partial x_2}, f_2\right) = \mathrm{resultant}\left(\frac{\partial f_2}{\partial x_2}, f_2, x_2\right) = \mathrm{dis}(f_2, x_2) = \mathrm{CP}_{f_2},$$

and the TSA is regular, we have that $\frac{\partial f_2}{\partial x_2} \neq 0$ when x_1 is on $[a,b]$ and $f_2 = 0$. Now, if

$$\mathrm{resultant}\left(\mathrm{res}\left(\frac{\partial f_2}{\partial x_1}, f_2\right), f_1(x_1), x_1\right) \neq 0,$$

then we can let $[a,b]$ be small enough so that $\mathrm{res}\left(\frac{\partial f_2}{\partial x_1}, f_2\right)$ has no zeros on $[a,b]$. That is to say, $[a,b]$ can be small enough so that x_2, regarded as a function of x_1 implicitly defined by f_2, is monotonic when x_1 is on $[a,b]$. Therefore, we can get the isolating intervals of $f_2(x^{(0)}, x_2)$ by making use of those isolating intervals of $f_2(a, x_2)$ and $f_2(b, x_2)$. More generally, we have the following definitions and algorithms.

Given a regular TSA T, for $2 \leq i \leq s, 1 \leq j < i$, let

$$U_{ij} = \begin{cases} \mathrm{res}\left(\frac{\partial f_i}{\partial x_j}, f_i, f_{i-1}, \ldots, f_{j+1}\right), & \text{if } \frac{\partial f_i}{\partial x_j} \not\equiv 0, \\ 1, & \text{if } \frac{\partial f_i}{\partial x_j} \equiv 0, \end{cases}$$

$$\mathrm{MP}_T(x_j) = \prod_{j \leq k < i \leq s} U_{ik}, (1 \leq j \leq s-1).$$

Algorithm: `REALZERO`

Input: a regular TSA $T^{(1)}$ and an optional parameter, w, indicating the maximal sizes of the output intervals on $x_1, \ldots, x_s$;

Output: isolating intervals of real solutions of $T^{(1)}$ or reports fail.

Step 1. $i \leftarrow 1$;
IF $\mathrm{resultant}(f_i(x_i), \mathrm{MP}_{T^{(i)}}(x_i), x_i) = 0$ THEN return(fail)
ELSE
$S^{(i)} \leftarrow \mathtt{nearzero}(f_i(x_i), \mathrm{CP}_{T^{(i)}} \cdot \mathrm{MP}_{T^{(i)}}, x_i)$
END IF

Step 2. FOR I in $S^{(i)}$ DO

Step 2a. Suppose $I = [a^{(1)}, b^{(1)}] \times \cdots \times [a^{(i)}, b^{(i)}]$ and let

$$V_I = \{(v^{(1)}, \ldots, v^{(i)}) | v^{(j)} \text{ is either } a^{(j)} \text{ or } b^{(j)} (1 \leq j \leq i)\}$$

be the set of the vertices of the i-dimensional cube I.

Step 2b. Let $1 \le j \le |V_I|$;

FOR $(v_j^{(1)}, \dots, v_j^{(i)})$ in V_I DO

substitute $x_1 = v_j^{(1)}, \dots, x_i = v_j^{(i)}$ into $T^{(1)}$;

delete the first i equations;

denote the other equations still by f_l $(i+1 \le l \le s)$;

denote the new system by $T_j^{(i+1)}$;

IF resultant$(f_{i+1}(x_{i+1}), \mathrm{MP}_{T_j^{(i+1)}}(x_{i+1}), x_{i+1}) = 0$ THEN

return(fail)

ELSE

$R_j^{(i+1)} \leftarrow$ `nearzero`$(f_{i+1}(x_{i+1}), \mathrm{CP}_{T_j^{(i+1)}} \cdot \mathrm{MP}_{T_j^{(i+1)}}, x_{i+1})$

END IF

END DO

Step 2c. Suppose $R_j^{(i+1)}$ is $\big[\alpha_{j,1}^{(i+1)}, \beta_{j,1}^{(i+1)}\big], \dots, \big[\alpha_{j,n_{i+1}}^{(i+1)}, \beta_{j,n_{i+1}}^{(i+1)}\big]$, define

$$R^{(i+1)} : \big[\alpha_1^{(i+1)}, \beta_1^{(i+1)}\big], \dots, \big[\alpha_{n_{i+1}}^{(i+1)}, \beta_{n_{i+1}}^{(i+1)}\big],$$

where for $1 \le k \le n_{i+1}$,

$$\alpha_k^{(i+1)} = \min\big(\alpha_{1,k}^{(i+1)}, \dots, \alpha_{|V_I|,k}^{(i+1)}\big), \beta_k^{(i+1)} = \max\big(\beta_{1,k}^{(i+1)}, \dots, \beta_{|V_I|,k}^{(i+1)}\big);$$

IF any two intervals in $R^{(i+1)}$ intersect OR

the maximal size of these intervals is greater than w THEN

$I \leftarrow$ `SHR`(I); # `SHR` is a subalgorithm given below

go back to Step 2a

ELSE

$S_I^{(i+1)} \leftarrow I \times R^{(i+1)}$

END IF

END DO (**Step 2**)

Step 3. $S^{(i+1)} \leftarrow \bigcup_{I \in S^{(i)}} S_I^{(i+1)}$, $i \leftarrow i+1$; If $i < s$, then go to Step 2.

Step 4. For each s-dimensional cube $I = [a^{(1)}, b^{(1)}] \times \cdots \times [a^{(s)}, b^{(s)}]$ in $S^{(s)}$, substitute $x_1 = a^{(1)}, \dots, x_s = a^{(s)}$ into each $g_j (1 \le j \le t)$ and check whether $g_j > 0$ or not. If all the inequalities are satisfied, output I.

Subalgorithm: `SHR`

Input: a k-dimensional cube I_0 in $S^{(k)}$;

Output: a k-dimensional cube $I \subset I_0$.

Step 0. Suppose $I_0 = [a_1, b_1] \times \cdots \times [a_k, b_k]$ and x_1^0 is the unique zero of $f_1(x_1)$ in $[a_1, b_1]$. By the intermediate value theorem, we can get an interval $[a_1', b_1'] \subset [a_1, b_1]$ with $x_1^0 \in [a_1', b_1']$ and $b_1' - a_1' = (b_1 - a_1)/10$.

Step 1. $i \leftarrow 1$, $I \leftarrow [a_1', b_1']$.

Step 2. Let V_I be the set of the vertices of the i-dimensional cube I;

FOR $(v_j^{(1)}, \ldots, v_j^{(i)})$ in V_I DO

substitute $x_1 = v_j^{(1)}, \ldots, x_i = v_j^{(i)}$ into $T^{(1)}$;

delete the first i equations of it;

denote the new system by $T_j^{(i+1)}$;

$Q_j^{i+1} \leftarrow \texttt{nearzero}(f_{i+1}(x_{i+1}), \mathrm{CP}_{T_j^{(i+1)}} \cdot \mathrm{MP}_{T_j^{(i+1)}}, x_{i+1})$

END DO

When `nearzero` is called to compute Q_j^{i+1}, let the maximal size of the intervals be $\frac{1}{10}$ of that we used to compute R_j^{i+1} in `REALZERO`.

Step 3. Merge $Q_j^{(i+1)}(1 \leq j \leq |V_I|)$ into one sequence $Q^{(i+1)}$ by the same way as we construct $R^{(i+1)}$ in `REALZERO`. Of course we know $[a_{i+1}, b_{i+1}]$ should correspond to which interval in $Q^{(i+1)}$. Denote the interval by $[a'_{i+1}, b'_{i+1}]$.

Step 4. $I \leftarrow I \times [a'_{i+1}, b'_{i+1}]$, $i \leftarrow i + 1$. If $i = k$, output I; else go to Step 2.

Remark 5. The correctness of the algorithm `REALZERO` is implied by Theorem 2.1 and the discussions at the beginning of Section 3.

Remark 6. In the steps of `REALZERO`, calling $\texttt{nearzero}(f_i(x_i), \mathrm{CP} \cdot \mathrm{MP}, x_i)$ aims at getting the isolating intervals of $f_i(x_i)$ that have the following two properties. (1) The property stated in Theorem 2.1; (2) Every $x_j (j > i)$, when viewed as a function of x_i implicitly defined by f_j, is monotonic on each isolating interval. The first property is guaranteed by Theorem 2.1 because the TSA is regular but the second one is not guaranteed. So, in some cases the algorithm does not work. For example, in the case that some zero of $f_1(x_1)$ is an extreme point of x_2 that is viewed as a function of x_1 implicitly defined by f_2.

Remark 7. When `REALZERO` does not work, we have tried the following method. Let

$$x_1 = y_1, x_2 = y_1 + y_2, \ldots, x_s = y_1 + y_2 + \cdots + y_s;$$

then the original TSA T is transformed into a new TSA T' in variables $y_1, \ldots, y_s$ and we hope that `REALZERO` works on T'. It does work on some problems but the correctness of the method has not been proved yet.

We illustrate the algorithm `REALZERO` in detail by the following simple example which we encountered while solving a geometric constraint problem.

Example 1. Given a regular TSA,

$$T^{(1)}: \begin{cases} f_1 = 10x^2 - 1 = 0, \\ f_2 = -5y^2 + 5xy + 1 = 0, \\ f_3 = 30z^2 - 20(y+x)z + 10xy - 11 = 0, \\ x \geq 0, y \geq 0, \end{cases}$$

by `REALZERO`, we take the following steps to get the isolating intervals.

Step 1. $\mathrm{MP}_{T^{(1)}}(x) = (5x^2 + 22)(110x^2 + 529)$ and $\mathrm{CP}_{T^{(1)}}(x) = x(4 + 5x^2)(7 + 2x^2)$ up to some non-zero constants. Because

$$\mathrm{resultant}(f_1(x), \mathrm{MP}_{T^{(1)}}(x), x) \neq 0,$$

we get

$$S^{(1)} = \texttt{nearzero}(f_1(x), \mathrm{CP}_{T^{(1)}} \cdot \mathrm{MP}_{T^{(1)}}, x)$$
$$= \left[\left[\frac{-3}{8}, \frac{-5}{16}\right], \left[\frac{5}{16}, \frac{3}{8}\right]\right].$$

Obviously, the first interval need not be considered in the following. So $S^{(1)} = \left[\frac{5}{16}, \frac{3}{8}\right]$.

Step 2. $S^{(1)}$ has only one interval $I = \left[\frac{5}{16}, \frac{3}{8}\right]$.

Step 2a. $V_I = \{v_1^{(1)} = \frac{5}{16}, v_2^{(1)} = \frac{3}{8}\}$.

Step 2b. Substituting $x = v_1^{(1)} = \frac{5}{16}$ into $T^{(1)}$ and deleting f_1 from it, we get

$$T_1^{(2)} : \begin{cases} f_2 = 1 + \frac{25}{16}y - 5y^2 = 0, \\ f_3 = 30z^2 - (20y + \frac{25}{4})z + \frac{25}{8}y - 11 = 0, \\ y \geq 0. \end{cases}$$

Now $\mathrm{MP}_{T_1^{(2)}}(y) = -1$ and $\mathrm{CP}_{T_1^{(2)}}(y) = \left(\frac{4349}{1280} - \frac{5}{16}y + y^2\right)y$, by

$$\texttt{nearzero}(f_2(y), \mathrm{CP}_{T_1^{(2)}} \cdot \mathrm{MP}_{T_1^{(2)}}, y),$$

we get $R_1^{(2)} = \left[\left[\frac{-3}{8}, \frac{-5}{16}\right], \left[\frac{5}{8}, \frac{11}{16}\right]\right]$. Obviously, the first interval need not be considered in the following, so, $R_1^{(2)} = \left[\frac{5}{8}, \frac{11}{16}\right]$. Similarly, by substituting $x = v_2^{(1)} = \frac{3}{8}$ into $T^{(1)}$, we get $R_2^{(2)} = \left[\frac{5}{8}, \frac{11}{16}\right]$

Step 2c. Merge $R_1^{(2)}$ and $R_2^{(2)}$ into $R^{(2)} : \left[\frac{5}{8}, \frac{11}{16}\right]$ and let $S_I^{(2)} = I \times R^{(2)}$.

Step 3. Because $S^{(1)}$ has only one interval, we have

$$S^{(2)} = S_I^{(2)} = \left[\frac{5}{16}, \frac{3}{8}\right] \times \left[\frac{5}{8}, \frac{11}{16}\right].$$

Now, $i = 2 < s = 3$, so, repeat Step 2 for $S^{(2)}$.

Step 2a. $S^{(2)}$ has only one element $I = \left[\frac{5}{16}, \frac{3}{8}\right] \times \left[\frac{5}{8}, \frac{11}{16}\right]$ and $V_I = \big\{(v_1^{(1)}, v_1^{(2)}) = \left(\frac{5}{16}, \frac{5}{8}\right), (v_2^{(1)}, v_2^{(2)}) = \left(\frac{5}{16}, \frac{11}{16}\right), (v_3^{(1)}, v_3^{(2)}) = \left(\frac{3}{8}, \frac{5}{8}\right),$
$(v_4^{(1)}, v_4^{(2)}) = \left(\frac{3}{8}, \frac{11}{16}\right)\big\}$.

Step 2b. Substituting $x = v_1^{(1)} = \frac{5}{16}, y = v_1^{(2)} = \frac{5}{8}$ into $T^{(1)}$ and deleting f_1, f_2 from it, we get $T_1^{(3)} : \{f_3 = 640z^2 - 400z - 193 = 0\}$. Because this is the last equation in the ascending chain, we let $\mathrm{CP}_{T_1^{(3)}} \cdot \mathrm{MP}_{T_1^{(3)}} = 1$ and, by $\texttt{nearzero}(f_3(z), 1, z)$, get $R_1^{(3)} = [[-1, 0], [0, 1]]$. Similarly, we have $R_2^{(3)} = R_3^{(3)} = R_4^{(3)} = [[-1, 0], [0, 1]]$.

Step 2c. Merge $R_1^{(3)}, R_2^{(3)}, R_3^{(3)}$ and $R_4^{(3)}$ into $R^{(3)} : [[-1, 0], [0, 1]]$ and let $S_I^{(3)} = I \times R^{(3)}$.

Because $S^{(2)}$ has only one element, we have

$$S^{(3)} = S_I^{(3)} = \left[\left[\frac{5}{16}, \frac{3}{8}\right] \times \left[\frac{5}{8}, \frac{11}{16}\right] \times [-1, 0], \left[\frac{5}{16}, \frac{3}{8}\right] \times \left[\frac{5}{8}, \frac{11}{16}\right] \times [0, 1]\right].$$

Now, $i = 3 = s$, so, go to Step 4 and output

$$\left[\left[\left[\frac{5}{16},\frac{3}{8}\right],\left[\frac{5}{8},\frac{11}{16}\right],[-1,0]\right],\left[\left[\frac{5}{16},\frac{3}{8}\right],\left[\frac{5}{8},\frac{11}{16}\right],[0,1]\right]\right].$$

4. `Realzero` and Examples

Following the discussion at the beginning of Section 2, we propose a method for isolating the real solutions of general semi-algebraic systems SAS by combining the algorithm `REALZERO` with the Ritt–Wu method and the algorithm given in Theorem 2.4. Our method has been implemented as a Maple program `realzero`. In general, for a SAS, the computation of `realzero` consists of three main steps. First, by the Ritt–Wu method, transform the system of equations into one or more systems in triangular form. In our implementation, we use `wsolve` (D. K. Wang, 2000), a program which realizes Wu's method under Maple. Second, for each component, check whether it is a regular TSA and, if not, transform it into regular TSAs by Theorem 2.4. Third, apply `REALZERO` to each resulting regular TSA. In this section, we report some examples computed by our program `realzero`. The performance of the program is presented in the appendices.

There are three basic kinds of calling sequences for a SAS defined in Section 1:

`realzero`$([p_1,\dots,p_n],[q_1,\dots,q_r],[g_1,\dots,g_t],[h_1,\dots,h_m],[x_1,\dots,x_s]);$
`realzero`$([p_1,\dots,p_n],[q_1,\dots,q_r],[g_1,\dots,g_t],[h_1,\dots,h_m],[x_1,\dots,x_s],width);$
`realzero`$([p_1,\dots,p_n],[q_1,\dots,q_r],[g_1,\dots,g_t],[h_1,\dots,h_m],[x_1,\dots,x_s],[w_1,\dots,w_s]).$

The command `realzero` returns a list of isolating intervals for all real solutions of the input system or reports that the method does not work on some components. If the 6th parameter "width", a positive number, is given, the maximal size of the output intervals is less than or equal to this number. If the 6th parameter is a list of positive numbers, $[w_1,\dots,w_s]$, the maximal sizes of the output intervals on $x_1,\dots,x_s$ are less than or equal to $w_1,\dots,w_s$, respectively. If the 6th parameter is omitted, the most convenient width is used for each interval returned. That is to say, the isolating intervals for certain x_i are returned provided that they do not intersect with each other.

Example 2. (Chemical Reaction) Given polynomial equations

$$\begin{cases} h_1 = 2 - 7x_1 + x_1^2x_2 - \frac{1}{2}(x_3 - x_1) = 0,\\ h_2 = 6x_1 - x_1^2x_2 - 5(x_4 - x_2) = 0,\\ h_3 = 2 - 7x_3 + x_3^2x_4 - \frac{1}{2}(x_1 - x_3) = 0,\\ h_4 = 6x_3 - x_3^2x_4 + 1 + \frac{1}{2}(x_2 - x_4) = 0,\end{cases}$$

by calling `realzero` $([h_1,h_2,h_3,h_4],[\,],[\,],[\,],[x_1,x_2,x_3,x_4])$, we get the isolating intervals of the real solutions as follows:

$$\left[\left[\left[\frac{-3}{16},\frac{-11}{64}\right],\left[\frac{-32}{81},\frac{2944}{1089}\right],\left[\frac{3683}{576},\frac{923}{144}\right],\left[\frac{-50}{81},\frac{2702}{1089}\right]\right],\right.$$
$$\left[\left[\frac{47}{16},\frac{189}{64}\right],\left[\frac{694144}{321489},\frac{43168}{19881}\right],\left[\frac{1883}{576},\frac{473}{144}\right],\left[\frac{622702}{321489},\frac{38750}{19881}\right]\right],$$
$$\left[\left[\frac{409}{64},\frac{205}{32}\right],\left[\frac{72896}{75645},\frac{1454464}{1505529}\right],\left[\frac{-53}{288},\frac{-97}{576}\right],\left[\frac{56086}{75645},\frac{1119902}{1505529}\right]\right],$$
$$\left.\left[\left[\frac{3665}{64},\frac{1833}{32}\right],\left[\frac{3177664}{30239001},\frac{508288}{4835601}\right],\left[\frac{-14705}{288},\frac{-29401}{576}\right],\left[\frac{-3542114}{30239001},\frac{-566290}{4835601}\right]\right]\right].$$

Example 3. (Neural Network) Given system

$$\begin{cases} f_1 = 1 - cx - xy^2 - xz^2 = 0, \\ f_2 = 1 - cy - yx^2 - yz^2 = 0, \\ f_3 = 1 - cz - zx^2 - zy^2 = 0, \\ f_4 = 8c^6 + 378c^3 - 27 = 0, \\ c > 0, 1 - c > 0, \end{cases}$$

by calling `realzero` $([f_1, f_2, f_3, f_4], [\,], [c, 1-c], [\,], [c, x, y, z])$, we get the isolating intervals of the real solutions as follows:

$$\Bigg[\Bigg[\left[\frac{13}{32}, \frac{7}{16}\right], [0,1], [0,1], [0,1]\Bigg], \Bigg[\left[\frac{849}{2048}, \frac{425}{1024}\right], \left[\frac{9}{32}, \frac{5}{16}\right], \left[\frac{1049}{640}, \frac{253}{144}\right], \left[\frac{9}{32}, \frac{5}{16}\right]\Bigg],$$
$$\Bigg[\left[\frac{434849}{1048576}, \frac{869699}{2097152}\right], \left[\frac{1}{4}, \frac{1}{2}\right], \left[\frac{492469823809}{1936677404672}, \frac{3411372355465}{5872690397184}\right], \left[\frac{2800317}{2787974}, \frac{3693919}{2001540}\right]\Bigg],$$
$$\Bigg[\left[\frac{1739397}{4194304}, \frac{869699}{2097152}\right], \left[\frac{3}{2}, 2\right], \left[\frac{357754}{1918275}, \frac{9511251}{27213598}\right], \left[\frac{3836549}{16777216}, \frac{6803401}{18874368}\right]\Bigg]\Bigg].$$

Example 4. (Cyclic 5) Given polynomial equations

$$\begin{cases} p_1 = a + b + c + d + e = 0, \\ p_2 = ab + bc + cd + de + ea = 0, \\ p_3 = abc + bcd + cde + dea + eab = 0, \\ p_4 = abcd + bcde + cdea + deab + eabc = 0, \\ p_5 = abcde - 1 = 0, \end{cases}$$

by calling `realzero` $([p_1, p_2, p_3, p_4, p_5], [\,], [\,], [\,], [a, b, c, d, e])$, we get the isolating intervals of the real solutions as follows:

$$\Bigg[\Bigg[[1,1], [1,1], \left[\frac{-21}{8}, \frac{-83}{32}\right], \left[\frac{-23}{60}, \frac{-29}{76}\right], [1,1]\Bigg]$$
$$\Bigg[[1,1], [1,1], \left[\frac{-13}{32}, \frac{-3}{8}\right], \left[\frac{-11}{4}, \frac{-9}{4}\right], [1,1]\Bigg],$$
$$\Bigg[[1,1], [1,1], [1,1], \left[-3, \frac{-5}{2}\right], \left[\frac{-1}{2}, 0\right]\Bigg],$$
$$\Bigg[[1,1], [1,1], [1,1], \left[\frac{-1}{2}, 0\right], \left[-3, \frac{-5}{2}\right]\Bigg],$$
$$\Bigg[\left[\frac{-343151}{131072}, \frac{-171575}{65536}\right], [1,1], [1,1], [1,1], \left[\frac{-36047189}{94372766}, \frac{-18023533}{47186222}\right]\Bigg],$$
$$\Bigg[\left[\frac{-25033}{65536}, \frac{-50065}{131072}\right], [1,1], [1,1], [1,1], \left[\frac{-1133}{302}, \frac{-2389}{926}\right]\Bigg],$$
$$\Bigg[\left[\frac{-21447}{8192}, \frac{-42893}{16384}\right], \left[\frac{-622702}{1630255}, \frac{-1245370}{3260421}\right], [1,1], [1,1], [1,1]\Bigg],$$
$$\Bigg[\left[\frac{-6259}{16384}, \frac{-3129}{8192}\right], \left[\frac{-186}{5}, \frac{-110}{47}\right], [1,1], [1,1], [1,1]\Bigg],$$
$$\Bigg[[1,1], \left[\frac{-671}{256}, \frac{-335}{128}\right], \left[\frac{-1086}{2843}, \frac{-542}{1419}\right], [1,1], [1,1]\Bigg],$$

$$\left[[1,1], \left[\frac{-49}{128}, \frac{-97}{256}\right], \left[\frac{-30}{11}, \frac{-62}{27}\right], [1,1], [1,1]\right]\right].$$

Example 5. Given a system of polynomial equations (Takeuchi and Lu, 1995),

$$\begin{cases} p_1 = 2x_1(2 - x_1 - y_1) + x_2 - x_1 = 0, \\ p_2 = 2x_2(2 - x_2 - y_2) + x_1 - x_2 = 0, \\ p_3 = 2y_1(5 - x_1 - 2y_1) + y_2 - y_1 = 0, \\ p_4 = y_2(3 - 2x_2 - 4y_2) + y_1 - y_2 = 0, \end{cases}$$

find the isolating intervals of non-negative solutions of it. Call

$$\texttt{realzero}([p_1, p_2, p_3, p_4], [x_1, x_2, y_1, y_2], [\,], [\,], [x_1, x_2, y_1, y_2], 1/1000);$$

the output is

$$\left[\left[\left[\frac{123699}{262144}, \frac{151}{320}\right],\right.\right.$$
$$\left[\frac{15604750193840633515355762525347641882989981}{15429603258688008185068797668747034522695597}, \frac{25646736065207290639}{25350470632055620751}\right],$$
$$\left[\frac{31940045261606640254 9}{15210282379233372450 6}, \frac{64807714054451707909444009190671657811201765}{30859206517376016370137595337494069045391194}\right],$$
$$\left.\left[\frac{11766526981955972576 8}{16304965803039035040 1}, \frac{23867887436121200844755218097593146520662280}{33070540167780718023098481036025768815988257}\right]\right],$$
$$\left[[0,0], [0,0], [0,0], [0,0]\right], \left[[0,0], [0,0], \left[\frac{77397}{32768}, \frac{38699}{16384}\right], \left[\frac{283969593}{268435456}, \frac{71012665}{67108864}\right]\right],$$
$$\left.\left[[2,2], [2,2], [0,0], [0,0]\right]\right],$$

which means the system has four non-negative real solutions.

Example 6. This is a problem of solving geometric constraints: Are we able to construct a triangle with elements $a = 1, R = 1$ and $h_a = \frac{1}{10}$ where a, h_a and R denote the side-length, altitude, and circumradius, respectively?

The result given by Mitrinovic *et al.* (1989) says that there exists a triangle with elements a, R, h_a if and only if $R1 = 2R - a \geq 0$ and $R2 = 8Rh_a - 4h_a^2 - a^2 \geq 0$. From our study (Yang *et al.*, 2001), we know that the result is incorrect. We can also see this from the following computations. For $a = 1, R = 1, h_a = \frac{1}{10}$, we have $R1 > 0, R2 < 0$ and

$$\begin{cases} f_1 = 1/100 - 4s(s-1)(s-b)(s-c) = 0, \\ f_2 = 1/5 - bc = 0, \\ f_3 = 2s - 1 - b - c = 0, \\ b > 0, c > 0, b + c - 1 > 0, 1 + c - b > 0, 1 + b - c > 0, \end{cases}$$

where s is the half perimeter and b, c are the lengths of the other two sides, respectively. Calling

$$\texttt{realzero}([f_1, f_2, f_3], [\,], [b, c, b + c - 1, 1 + c - b, 1 + b - c], [\,], [s, b, c]);$$

we get

$$\left[\left[\left[\frac{259}{256}, \frac{519}{512}\right], \left[\frac{33}{128}, \frac{17}{64}\right], \left[\frac{97}{128}, \frac{197}{256}\right]\right], \left[\left[\frac{259}{256}, \frac{519}{512}\right], \left[\frac{97}{128}, \frac{99}{128}\right], \left[\frac{1}{4}, \frac{69}{256}\right]\right],\right.$$

$$\left[\left[\left[\frac{297}{256},\frac{595}{512}\right],\left[\frac{11}{64},\frac{23}{128}\right],\left[\frac{73}{64},\frac{295}{256}\right]\right],\left[\left[\frac{297}{256},\frac{595}{512}\right],\left[\frac{73}{64},\frac{37}{32}\right],\left[\frac{21}{128},\frac{47}{256}\right]\right]\right],$$

which means that there are two different triangles with elements $a = 1, R = 1$ and $h_a = \frac{1}{10}$ since b and c are symmetric in the system.

Example 7. Folke (1994) asked for the solution of the following problem: Which triangles can occur as sections of a regular tetrahedron by planes which separate one vertex from the other three? In fact, this is one of the special cases of the p-3-p problem which originates from camera calibration. Making use of another program called "DISCOVERER" (Yang *et al.*, 2001), we have got the so-called complete solution classification of this problem.

Now, let $1, a, b$ be the lengths of three sides of the triangle (assume $b \geq a \geq 1$), and x, y, z the distances from the vertex to the three vertexes of the triangle respectively and suppose that (a, b) is the real roots of $\{a^2-1+b-b^2 = 0, 3b^6+56b^4-122b^3+56b^2+3 = 0\}$. We want to find x, y and z. Thus, the system is

$$\begin{cases} h_1 = x^2 + y^2 - xy - 1 = 0, \\ h_2 = y^2 + z^2 - yz - a^2 = 0, \\ h_3 = z^2 + x^2 - zx - b^2 = 0, \\ h_4 = a^2 - 1 + b - b^2 = 0, \\ h_5 = 3b^6 + 56b^4 - 122b^3 + 56b^2 + 3 = 0, \\ x > 0, y > 0, z > 0, a - 1 \geq 0, b - a \geq 0, a + 1 - b > 0. \end{cases}$$

Call

$$\texttt{realzero}([h_1, h_2, h_3, h_4, h_5], [b - a, a - 1], [x, y, z, a + 1 - b], [\,], [b, a, x, y, z]);$$

the output is

$$\left[\left[\left[\frac{162993}{131072},\frac{81497}{65536}\right],\left[\frac{73}{64},\frac{147}{128}\right],\left[\frac{1181}{1024},\frac{2363}{2048}\right],\left[\frac{1349206836}{2188300897},\frac{348432792}{556866289}\right],\right.\right.$$
$$\left.\left.\left[\frac{3247431090114025}{2465566125550592},\frac{2029443732706441}{1540423210501112}\right]\right]\right].$$

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Appendix A. Timing Data

All the examples were computed on a PC (Pentium 933 MHz CPU, 128 MB of main memory) with Maple 5.4.

Time unit: s

Example No.	Ex.2	Ex.3	Ex.4	Ex.5	Ex.6	Ex.7
Triangular form	0.107	0.623	2.011	0.384	0.014	0.137
Regular TSA	0.005	1.578	2.772	2.682	0.015	1.774
REALZERO	0.396	15.382	2.889	3.07	0.45	33.840
Total time	0.508	17.583	7.672	6.136	0.479	35.751

Appendix B. Resulting Regular TSAs

Ex.2. $\{81x_1^4 - 5382x_1^3 + 43960x_1^2 - 79632x_1 - 15680 = 0, -10 - 54x_1 + 9x_1^2x_2 = 0, -56 + 9x_1 + 9x_3 = 0, 9x_1^2x_4 + 2x_1^2 - 10 - 54x_1 = 0\}$.

Ex.3. $\{f_4=0, 2x^3+cx-1=0, x-y=0, x-z=0, c>0, 1-c>0\}$,
$\{f_4=0, 1+x^3+cx=0, x^2y^2+cx^2+cy^2-y+c^2-x=0, c+x^2-zy=0, c>0, 1-c>0\}$,
$\{f_4=0, -16x^3c^9-144x^3c^6-432x^3c^3-432x^3-88x^2c^8-528c^5x^2-792c^2x^2-24xc^{10}-700xc^7-2100xc^4-648cx-124c^9-3382c^6-972c^3+216=0, -xy+c+x^2=0, x-z=0, c>0, 1-c>0\}$,
$\{f_4=0, -16x^3c^9-144x^3c^6-432x^3c^3-432x^3-88x^2c^8-528c^5x^2-792c^2x^2-24xc^{10}-700xc^7-2100xc^4-648cx-124c^9-3382c^6-972c^3+216=0, 2x^3y+cx^2+2xcy-2y+c^2+x=0, x^3+zx^2+cx-1+cz=0, c>0, 1-c>0\}$,
$\{f_4=0, -16x^3c^28-1992x^3c^{25}-82668x^3c^{22}-1143574x^3c^{19}+32x^2c^{27}+3872x^2c^{24}+155968x^2c^{21}+2088280x^2c^{18}-124002x^2c^{15}-8xc^{29}-1028xc^{26}-44982xc^{23}-709803xc^{20}-1724088xc^{17}+329850xc^{14}-13446xc^{11}-8c^{28}-1020c^{25}-42594c^{22}-549589c^{19}+1359495c^{16}-461970c^{13}+46980c^{10}-1458c^7=0, xcy-x+cx^2+y=0, -cx+2zx^2-1=0, c>0, 1-c>0\}$,
where $f_4=8c^6+378c^3-27$.

Ex.4. $\{1+a+a^2+a^3+a^4=0, b-1=0, ca-1=0, a^2+da+a+d+1=0, a^2e+a^2+ea+1+a=0\}$,
$\{a^4+a^3+6a^2-4a+1=0, 129a-18b-170a^2-102a^3+89ab+89a^3b-178a^2b-34=0, 885a-123c-699a^3+610a^3c-1165a^2-1220a^2c+610ca-233=0, -65+322a^3+170a^3d-644a^2+1013da^2+322a-666da+163d=0, 25-123a^3+170a^3e+246a^2+1013a^2e-123a-666ea+163e=0\}$,
$\{1+a+a^2+a^3+a^4=0, a^3+a^2+a^2b+a+ab^2+ab+1+b^3+b^2+b=0, ca-b^2=0, a^2+da+ab+db+b^2=0, -a^2e-eab+a^3+a^2+a+ab+1+b^2+b=0\}$,
$\{a-1=0, b^2+3b+1=0, 2b+2c+1+5bc=0, -1+d=0, e-1=0\}$,
$\{1+a+a^2+a^3+a^4=0, b^2+3ab+a^2=0, -a-2b-2c+5bc+5abc+5a^2bc+5a^3bc=0, -d+a=0, a-e=0\}$,
$\{a-1=0, b-1=0, c^2+3c+1=0, 4cd+d+c-1=0, e-1=0\}$,
$\{1+a+a^2+a^3+a^4=0, a-b=0, a-c=0, d^2+3da+a^2=0, 3a+d+e=0\}$,
$\{a^4-4a^3+6a^2+a+1=0, 36a-71b+169a^2-39a^3-89ab+178a^3b-445a^2b+26=0, -249a-186c-1157a^2+267a^3-233ca+466a^3c-1165a^2c-178=0, -61d-82a-377a^2+87a^3-76da+152a^3d-380da^2-58=0, -61e-82a-377a^2+87a^3-76ea+152a^3e-380a^2e-58=0\}$,
$\{1+a+a^2+a^3+a^4=0, a-b=0, 3ca+a^2+c^2=0, -da-ca-4cd+a^2=0, a-e=0\}$,
$\{a-1=0, b-1=0, -1+c=0, d^2+3d+1=0, d+e+3=0\}$,
$\{a-1=0, b^4+b^3+b^2+b+1=0, b^2-c=0, db+b^2+d+b+1=0, b^3+b^2+eb+b+e=0\}$,
$\{a^2+3a+1=0, 34a+34b+89ab+13=0, -1+c=0, -1+d=0, e-1=0\}$,
$\{a^2+3a+1=0, b-1=0, -1+c=0, -1+d=0, 47+123a+322ea+123e=0\}$,
$\{a^4+a^3+6a^2-4a+1=0, -49a-18b+65a^2+39a^3+89ab+89a^3b-178a^2b+13=0, 338a-47c-445a^2-267a^3+233ca+233a^3c-466a^2c-89=0, -15d+111a-145a^2-87a^3+76da+76a^3d-152da^2-29=0, -15e+111a-145a^2-87a^3+76ea+76a^3e-152a^2e-29=0\}$,
$\{a^4-4a^3+6a^2+a+1=0, -95a-71b-442a^2+102a^3-89ab+178a^3b-445a^2b-68=0, -652a-487c+699a^3+1220a^3c-3029a^2-3050a^2c-610ca-466=0, -257+644a^3+673a^3d-1610a^2-1006da^2-322a-177da-170d, 98-246a^3+673a^3e+615a^2-1006a^2e+123a-177ea-170e=0\}$.

Ex.5. $\{x_1=0, x_2=0, y_1=0, y_2=0\}$,
$\{x_1=0, x_2=0, 17+316y_1-288y_1^2+64y_1^3=0, 9y_1-4y_1^2+y_2=0, y_1>0, y_2>0\}$
$\{512x_1^9+1536x_1^8-3456x_1^7-12544x_1^6+6320x_1^5+33672x_1^4-2728x_1^3-32874x_1^2+$

$450x_1+5729=0, -36130x_1^2-714x_1+57160x_2+5729-36620x_2x_1-302056x_1^2x_2+204128x_1^4x_2+194072x_1^3x_2+4544x_1^3+43848x_1^4-60736x_1^6x_2-13632x_1^6-9296x_1^5-93664x_1^5x_2+12544x_1^7x_2+1536x_1^8+6656x_1^8x_2+1408x_1^7=0, 847462x_1^2-36620x_1y_1+104612x_1-160799-552612x_1^3-557664x_1^4+162944x_1^6+261920x_1^5-17664x_1^8+57160y_1+12544x_1^7y_1+6656x_1^8y_1+194072x_1^3y_1-302056x_1^2y_1-93664x_1^5y_1+204128x_1^4y_1-60736x_1^6y_1-34624x_1^7=0, 582y_2+81640x_1^2+7319x_1+1628y_2x_1-40706x_1^3-64256x_1^4+19600x_1^6+22472x_1^5-2048x_1^8-5088x_1^3y_2-3328x_1^7-15163+7808x_1^4y_2+256x_1^8y_2-2432x_1^6y_2+544x_1^5y_2-3636x_1^2y_2=0, x_1>0, x_2>0, y_1>0, y_2>0\}$,

$\{x_1-2=0, x_2-2=0, y_1=0, y_2=0, x_1>0, x_2>0\}$,

$\{2x_1^2-2x_1-1=0, 19x_2+74x_2x_1+19x_1+18=0, y_1=0, y_2=0, x_1>0, x_2>0\}$.

Ex.6. $\{1+400s^4-800s^3+320s^2+80s=0, 1-10sb+5b+5b^2=0, -2s+1+b+c=0, b>0, c>0, b+c-1>0, 1+c-b>0, 1+b-c>0\}$.

Ex.7. $\{3b^6+56b^4-122b^3+56b^2+3=0, a^2-1+b-b^2=0, 36-792b-72x^2+144x^4+680640x^2b^6+459277260x^2b^{10}-185266632x^2b^9-11006916x^2b^7+1269185400x^2b^{12}+55315614x^2b^8+9417b^2+1224x^2b-80223b^3-10482x^2b^2+13419746b^6+536785b^{24}-80223b^{25}+36b^{28}+9417b^{26}-792b^{27}-2936934b^{23}-51728470b^{21}+13419746b^{22}+536785b^4+856889496x^4b^{10}-1191653856x^4b^{11}+1330506792x^4b^{12}-2936934b^5-7235784x^4b^5-257112x^4b^3-496127088x^4b^9+1526652x^4b^4+27960660x^4b^6+232553952x^4b^8-88940736x^4b^7-868818312x^2b^{11}+1269185400b^{14}x^2-1191653856b^{13}x^4-1439332416b^{13}x^2-868818312b^{15}x^2+55315614x^2b^{18}-185266632x^2b^{17}+856889496b^{14}x^4+459277260b^{16}x^2-10482x^2b^{24}+1224x^2b^{25}-72x^2b^{26}+59724x^2b^{23}+479520x^2b^{21}-230760x^2b^{22}+59724x^2b^3+33300x^4b^2-3024x^4b+1080244823b^{18}+3384674734b^{12}-2097162390b^{11}+1080244823b^{10}-465418571b^9-2097162390b^{17}+3384674734b^{16}-4519543020b^{15}+4978448892b^{14}-4519543020b^{13}+168760413b^8-51728470b^7-465418571b^{19}+168760413b^{20}-230760b^4x^2+479520b^5x^2+144b^{24}x^4-11006916x^2b^{19}+680640x^2b^{20}-3024b^{23}x^4+33300b^{22}x^4-257112b^{21}x^4-496127088b^{15}x^4+27960660b^{18}x^4+1526652b^{20}x^4-7235784b^{19}x^4-88940736b^{17}x^4+232553952b^{16}x^4=0, -2yx^2+yb^2+2x^3-3xb+yb+x=0, 2x^4-x^2b-2x^2b^2+b^2zx+b^3-2xbz+b^2-x^2+zx=0, b-a>0, a-1>0, a+1-b>0, x>0, y>0, z>0\}$.

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