CHAPTER 1

AUTOMATED DEDUCTION IN REAL GEOMETRY

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Including three aspects, problem solving, theorem proving and theorem discovering, automated deduction in real geometry essentially depends upon the semi-algebraic system solving. A "semi-algebraic system" is a system consisting of polynomial equations, polynomial inequations and polynomial inequalities, where all the polynomials are of integer coefficients. We give three practical algorithms for the above three kinds of problems, respectively. A package based on the three algorithms for "solving" semi-algebraic systems at each of the three levels has been implemented as Maple programs. The performance of the package on many famous examples are reported.

1. Introduction

A semi-algebraic system is a system of polynomial equations, inequalities and inequations. More precisely, we call

$$\begin{cases} p_1(x_1,...,x_n)=0,...,p_s(x_1,...,x_n)=0,\\ g_1(x_1,...,x_n)\ge 0,...,g_r(x_1,...,x_n)\ge 0,\\ g_{r+1}(x_1,...,x_n)>0,...,g_t(x_1,...,x_n)>0,\\ h_1(x_1,...,x_n)\ne 0,...,h_m(x_1,...,x_n)\ne 0, \end{cases} \tag{1}$$

a *semi-algebraic system* (SAS for short), where $n, s \geq 1,\ r, t, m \geq 0$ and $p_i,\ g_j,\ h_k$ are all polynomials in $x_1, ..., x_n$ with integer coefficients.

Many problems in both practice and theory can be reduced to problems of solving SAS. For example, we may mention some special cases of the "p-3-p" problem[15] which originates from computer vision, the problem of constructing limit cycles for plane differential systems[26] and the problem of automated discovering and proving for geometric inequalities[49,48]. Moreover, many problems in geometry, topology and differential dynamical systems are expected to be solved by translating them into certain semi-algebraic systems.

There are two classical methods, Tarski's method[32] and the cylindrical algebraic decomposition (CAD) method proposed by Collins[10], for solving semi-algebraic systems and numerous improvements and progresses[11,7,14,3] have been made since then. But this problem is well-known to have for general case double exponential complexity in the number of variables[13]. Therefore, the best way to attack quantifier elimination may be that to classify the problems and to offer practical algorithms for some special cases from various applications[36,37,38,19,16,48,49,52].

Two classes of SASs with strong geometric backgrounds are discussed in this paper. A SAS is called a *constant-coefficient* SAS if $n = s$ and $\{p_1, ..., p_s\}$ is assumed to have only a finite number of common zeros while a SAS is called a *parametric* SAS if $s < n$ (s indeterminates are viewed as variables and the other $n - s$ indeterminates parameter) and $\{p_1, ..., p_s\}$ is assumed to have only a finite number of common zeros on all the possible values of the parameter. A very recent algorithm to solve general SAS (the ideal generated by the polynomials may be of positive dimension) appears in the recent paper by P. Aubry *et al.*[2].

For a constant-coefficient SAS, counting and isolating real solutions are two key problems in the study of the real solutions of the system from the viewpoint of symbolic computation. And algorithms for this kind of problems often form the base of some other algorithms for solving parametric SASs. T. Becker and V. Weispfenning[4] presented an algorithm for isolating the real zeros of a system of polynomial equations by Gröbner bases computing and Sturm theorem. Some effective methods for counting real solutions of a SAS are those using trace forms or the rational univariate representation[28,29,17] and the algorithm proposed by Xia and Hou[44]. Usually, these methods may suggest some algorithms for isolating real solutions of a SAS. In Section 2, we present an algorithm[45] for isolating the real solutions of a constant-coefficient SAS, which, in some sense, can be viewed as a generalization of the Uspensky algorithm[12]. Lu *et al.*[25] proposed a different algorithm for isolating the real solutions of polynomial equations. Recently, Xia and Zhang[46] presented a new and faster algorithm for isolating the real zeros of polynomial equations based on interval arithmetic.

Sections 3 and 4 are devoted to algorithms for "solving" parametric SASs. Automated theorem proving and discovering on inequalities are always considered as difficult topics in the area of automated reasoning. To prove or disprove a geometric inequality, it is often required to decide whether a parametric SAS has any real solutions or not. A so-called "dimension-decreasing" algorithm[52,51] is very fast for this kind of problems and is sketched in Section 3. To discover inequality-type theorems automatically, it is often required to find conditions on the parameter of a parametric SAS such that the system has a specified number of real solutions. A complete and practical algorithm for this kind of problems is described in Section 4.

2. Find Real Solutions of Geometric Problems

In this section we discuss an algorithm for isolating the real solutions of a constant-coefficient SAS and its application to finding real solutions of geometric problems.

2.1. *Basic Definitions*

For any polynomial P with positive degree, the *leading variable* x_l of P is the one with greatest index l that effectively appears in P. A *triangular set* is a set of polynomials $\{f_i(x_1,...,x_i), f_{i+1}(x_1,...,x_{i+1}),...,f_l(x_1,...,x_l)\}$ in which the leading variable of f_j is x_j. If the ideal generated by $p_1,...,p_n$ is zero dimensional, then it is well known that the Ritt-Wu method, Gröbner basis methods or subresultant methods can be used to transform the system of equations into one or more systems in triangular form[41,8,34,1,54]. Therefore, in this section, we only consider triangular sets and the problem we discuss is to isolate the real solutions of the following system

$$\begin{cases} f_1(x_1) = 0, \\ f_2(x_1, x_2) = 0, \\ \cdots\cdots \\ f_s(x_1, x_2, ..., x_s) = 0, \\ g_1(x_1, x_2, ..., x_s) \geq 0, ..., g_r(x_1, x_2, ..., x_s) \geq 0, \\ g_{r+1}(x_1, x_2, ..., x_s) > 0, ..., g_t(x_1, x_2, ..., x_s) > 0, \\ h_1(x_1, x_2, ..., x_s) \neq 0, ..., h_m(x_1, x_2, ..., x_s) \neq 0, \end{cases} \tag{2}$$

where $s \geq 1,\ r, t, m \geq 0$ and $\{f_1, f_2, ..., f_s\}$ is a normal ascending chain[54] (also see Definition 1 in this section). We call a system in this form a *triangular semi-algebraic system* (TSA for short).

Given a polynomial $g(x)$, let resultant(g, g'_x, x) be the Sylvester resultant of g and g'_x with respect to x, where g'_x means the derivative of $g(x)$ with respect to x. We call it the *discriminant* of g with respect to x and denote it by dis(g, x) or simply by dis(g) if its meaning is clear.

Given a polynomial g and a triangular set $\{f_1, f_2, ..., f_s\}$, let

$$r_s := g, \quad r_{s-i} := \mathrm{resultant}(r_{s-i+1}, f_{s-i+1}, x_{s-i+1}), \quad i = 1, 2, ..., s;$$

$$q_s := g, \quad q_{s-i} := \mathrm{prem}(q_{s-i+1}, f_{s-i+1}, x_{s-i+1}), \quad i = 1, 2, ..., s,$$

where resultant(p, q, x) means the Sylvester resultant of p, q with respect to x and prem(p, q, x) means the pseudo-remainder of p divided by q with respect to x.

We denote r_{i-1} and q_{i-1} $(1 \leq i \leq s)$ by res$(g, f_s, ..., f_i)$ and prem$(g, f_s, ..., f_i)$ and call them the *resultant* and *pseudo-remainder* of g with respect to the triangular set $\{f_i, f_{i+1}, ..., f_s\}$, respectively.

Definition 1: Given a triangular set $\{f_1, f_2, ..., f_s\}$, denote by I_i $(i = 1, ..., s)$ the leading coefficient of f_i in x_i. A triangular set $\{f_1, f_2, ..., f_s\}$ is called a *normal ascending chain* if res$(I_i, f_{i-1}, ..., f_1) \neq 0$ for $i = 2, ..., s$. Note that $I_1 \neq 0$ follows from the definition of a triangular set.

Remark 2: A normal ascending chain is also called a *regular chain* by Kalkbrener[21] and a *regular set* by Wang[35], and was called a *proper ascending chain* by Yang and Zhang[53].

Definition 3: Let a TSA be given as defined in (2), called T. For every f_i $(i \geq 1)$, let $\mathrm{CP}_{f_i} = \mathrm{dis}(f_i, x_i)$ $(i \leq 2)$ and

$$\mathrm{CP}_{f_i} = \mathrm{res}\,(\mathrm{dis}(f_i, x_i), f_{i-1}, f_{i-2}, ..., f_2), \;\; i > 2.$$

For any $q \in \{g_j \mid 1 \leq j \leq t\} \bigcup \{h_k \mid 1 \leq k \leq m\}$, let

$$\mathrm{CP}_q = \begin{cases} \mathrm{res}\,(q, f_s, f_{s-1}, ..., f_2), & \text{if } s > 1, \\ q, & \text{if } s = 1. \end{cases}$$

We define

$$\mathrm{CP}_T(x_1) = \prod_{1 \leq i \leq s} \mathrm{CP}_{f_i} \cdot \prod_{1 \leq j \leq t} \mathrm{CP}_{g_j} \cdot \prod_{1 \leq k \leq m} \mathrm{CP}_{h_k},$$

and call it the *critical polynomial* of the system T with respect to x_1. We also denote $\mathrm{CP}_T(x_1)$ by CP or $\mathrm{CP}(x_1)$ if its meaning is clear.

Remark 4: Let a TSA T be given and denote by T_1 the system formed by deleting $f_1(x_1)$ from T. In T_1, we view x_1 as a parameter and let it vary continuously on the real number axis. From Theorem 7 below, we know that the number of distinct real solutions of T_1 will remain fixed provided that x_1 varies on an interval in which there are no real zeros of $\mathrm{CP}_T(x_1)$. That is why $\mathrm{CP}_T(x_1)$ is called the *critical polynomial* of the system T.

Definition 5: A TSA is *regular* if $\mathrm{resultant}(f_1(x_1), \mathrm{CP}(x_1), x_1) \neq 0$.

Remark 6: According to Definition 5, for a regular TSA no CP_{h_k} $(1 \leq k \leq m)$ has common zeros with $f_1(x_1)$, which implies that every solution of $\{f_1 = 0, ..., f_s = 0\}$ satisfies $h_k \neq 0$ $(1 \leq k \leq m)$. Thus if a TSA is regular we can omit the h_k's in it without loss of generality. Similarly, every solution of $\{f_1 = 0, ..., f_s = 0\}$ satisfies $g_j \neq 0$ $(1 \leq j \leq t)$. That is to say, each of the inequalities $g_j \geq 0$ $(1 \leq j \leq r)$ in a regular TSA can be treated as $g_j > 0$.

2.2. *The Algorithm*

Given two polynomials $p(x)$, $q(x) \in \mathbb{Z}[x]$, suppose $p(x)$ and $q(x)$ have no common zeros, i.e., $\mathrm{resultant}(p, q, x) \neq 0$, and $\alpha_1 < \alpha_2 < ... < \alpha_n$ are all distinct real zeros of $p(x)$. By the modified Uspensky algorithm[12,30], we can obtain a sequence of intervals, $[a_1, b_1], ..., [a_n, b_n]$, satisfying

1) $\alpha_i \in [a_i, b_i]$ for $i = 1, ..., n$,
2) $[a_i, b_i] \bigcap [a_j, b_j] = \emptyset$ for $i \neq j$,
3) a_i, b_i $(1 \leq i \leq n)$ are all rational numbers, and
4) the maximal size of each isolating interval can be less than any positive number given in advance.

Because $p(x)$ and $q(x)$ have no common zeros, the intervals can also satisfy

5) no zeros of $q(x)$ are in any $[a_i, b_i]$.

In the following we denote an algorithm to do this by `nearzero`(p, q, x), or `nearzero`(p, q, x, ϵ) if the maximal size of the isolating intervals is specified to be not greater than a positive number ϵ.

Theorem 7: Let a regular TSA be given. Suppose $f_1(x_1)$ has n distinct real zeros; then, by calling `nearzero`$(f_1, \mathrm{CP}(x_1), x_1)$ we can obtain a sequence of intervals, $[a_1, b_1], ..., [a_n, b_n]$, satisfying, for any $[a_i, b_i]$ $(1 \leq i \leq n)$ and any $\beta, \gamma \in [a_i, b_i]$,

1) if $s > 1$, then the system

$$\begin{cases} f_2(\beta, x_2) = 0, ..., f_s(\beta, x_2, ..., x_s) = 0, \\ g_1(\beta, x_2, ..., x_s) > 0, ..., g_t(\beta, x_2, ..., x_s) > 0, \end{cases}$$

and the system

$$\begin{cases} f_2(\gamma, x_2) = 0, ..., f_s(\gamma, x_2, ..., x_s) = 0, \\ g_1(\gamma, x_2, ..., x_s) > 0, ..., g_t(\gamma, x_2, ..., x_s) > 0, \end{cases}$$

have the same number of distinct real solutions and,

2) if $s = 1$, then for any g_j $(1 \le j \le t)$, $\text{sign}(g_j(\beta)) = \text{sign}(g_j(\gamma))$, where $\text{sign}(x)$ is 1 if $x > 0$, -1 if $x < 0$ and 0 if $x = 0$.

Theorem 8: [45] For an irregular TSA T, there is an algorithm which can decompose T into regular systems T_i. Let all the distinct real solutions of a given system be denoted by $Rzero(\cdot)$; then this decomposition satisfies $Rzero(T) = \bigcup Rzero(T_i)$.

By Theorem 8, we only need to consider regular TSAs. Given a regular TSA T, for $2 \le i \le s$, $1 \le j < i$, let

$$U_{ij} = \begin{cases} \text{res}\,(\dfrac{\partial f_i}{\partial x_j}, f_i, f_{i-1}, ..., f_{j+1}), & \text{if } \dfrac{\partial f_i}{\partial x_j} \not\equiv 0, \\ 1, & \text{if } \dfrac{\partial f_i}{\partial x_j} \equiv 0, \end{cases}$$

$$\text{MP}_T(x_j) = \prod_{j \le k < i \le s} U_{ik}, \ (1 \le j \le s-1).$$

Algorithm: `REALZERO`

Input: a regular TSA $T^{(1)}$ and an optional parameter, w, indicating the maximal sizes of the output intervals on $x_1, ..., x_s$;

Output: isolating intervals of real solutions of $T^{(1)}$ or reports fail.

Step 1. $i \leftarrow 1$; Compute $\text{resultant}(f_i(x_i), \text{MP}_{T^{(i)}}(x_i), x_i)$. If it is zero, then return "fail" and stop. Otherwise,

$$S^{(i)} \leftarrow \texttt{nearzero}(f_i(x_i), \text{CP}_{T^{(i)}} \cdot \text{MP}_{T^{(i)}}, x_i).$$

Step 2. For each i-dimensional cube I in $S^{(i)}$,

Step 2a. Let V_I be the set of the vertices of the i-dimensional cube I.

Step 2b. For each vertex $(v_j^{(1)}, ..., v_j^{(i)})$ in V_I, substitute $x_1 = v_j^{(1)}, ..., x_i = v_j^{(i)}$ into $T^{(1)}$ and delete the first i equations (denote the other equations still by f_l $(i+1 \leq l \leq s)$ and the new system by $T_j^{(i+1)}$). Compute

$$\text{resultant}(f_{i+1}(x_{i+1}), \text{MP}_{T_j^{(i+1)}}(x_{i+1}), x_{i+1}).$$

If it is zero, return "fail" and stop. Otherwise,

$$R_j^{(i+1)} \leftarrow \texttt{nearzero}(f_{i+1}(x_{i+1}), \text{CP}_{T_j^{(i+1)}} \cdot \text{MP}_{T_j^{(i+1)}}, x_{i+1}).$$

Step 2c. Merge all $R_j^{(i+1)}$ into one list of intervals, denoted by $R^{(i+1)}$. If any two intervals in $R^{(i+1)}$ intersect or the maximal size of these intervals is greater than w, shrink I by a sub-algorithm $\texttt{SHR}(I)$ given below and go back to Step 2a. Otherwise,

$$S_I^{(i+1)} \leftarrow I \times R^{(i+1)}.$$

Step 3. $S^{(i+1)} \leftarrow \bigcup_{I \in S^{(i)}} S_I^{(i+1)}, \quad i \leftarrow i+1$; If $i < s$, then go to Step 2.

Step 4. For each s-dimensional cube I, check the sign of each g_j $(1 \leq j \leq t)$ on I and determine the output.

Sub-algorithm: SHR

Input: a k-dimensional cube I_0 in $S^{(k)}$;

Output: a k-dimensional cube $I \subset I_0$.

Step 0. Suppose $I_0 = [a_1, b_1] \times \cdots \times [a_k, b_k]$ and x_1^0 is the unique zero of $f_1(x_1)$ in $[a_1, b_1]$. By the intermediate value theorem, we can get an interval $[a_1^{'}, b_1^{'}] \subset [a_1, b_1]$ with $x_1^0 \in [a_1^{'}, b_1^{'}]$ and $b_1^{'} - a_1^{'} = (b_1 - a_1)/8$.

Step 1. $i \leftarrow 1, \quad I \leftarrow [a_1^{'}, b_1^{'}]$.

Step 2. Let V_I be the set of the vertices of the i-dimensional cube I. For each $(v_j^{(1)}, ..., v_j^{(i)})$ in V_I, substitute $x_1 = v_j^{(1)}, ..., x_i = v_j^{(i)}$ into $T^{(1)}$ and delete the first i equations of it (denote the new system by $T_j^{(i+1)}$).

$$Q_j^{i+1} \leftarrow \texttt{nearzero}\,(f_{i+1}(x_{i+1}), \text{CP}_{T_j^{(i+1)}} \cdot \text{MP}_{T_j^{(i+1)}}, x_{i+1})$$

When nearzero is called to compute Q_j^{i+1}, let the maximal size of the intervals be $\dfrac{1}{8}$ of that we used to compute R_j^{i+1} in REALZERO.

Step 3. Merge $Q_j^{(i+1)}$ into one sequence $Q^{(i+1)}$. Of course we know $[a_{i+1}, b_{i+1}]$ should correspond to which interval in $Q^{(i+1)}$. Denote the interval by $[a_{i+1}^{'}, b_{i+1}^{'}]$.

Step 4. $I \leftarrow I \times [a_{i+1}^{'}, b_{i+1}^{'}], \quad i \leftarrow i+1$. If $i = k$, output I and stop. otherwise, go to Step 2.

Remark 9: In the steps of REALZERO, calling nearzero$(f_i(x_i), \mathrm{CP}\cdot\mathrm{MP}, x_i)$ aims at getting the isolating intervals of $f_i(x_i)$ that have the following two properties. (1). The property stated in Theorem 7; (2). Every x_j $(j > i)$, when viewed as a function of x_i implicitly defined by f_j, is monotonic on each isolating interval. The first property is guaranteed by Theorem 7 because the TSA is regular but the second one is not guaranteed. So, in some cases the algorithm does not work. For example, in the case that some zero of $f_1(x_1)$ is an extreme point of x_2 that is viewed as a function of x_1 implicitly defined by f_2.

We illustrate the algorithm REALZERO in detail by the following simple example which we encountered while solving a geometric constraint problem.

Example 10: Given a regular TSA

$$T^{(1)}: \begin{cases} f_1 = 10x^2 - 1 = 0, \\ f_2 = -5y^2 + 5xy + 1 = 0, \\ f_3 = 30z^2 - 20(y+x)z + 10xy - 11 = 0, \\ x \geq 0,\ y \geq 0, \end{cases}$$

by REALZERO, we take the following steps to get the isolating intervals.

Step 1. $\mathrm{MP}_{T^{(1)}}(x) = (5x^2 + 22)(110x^2 + 529)$ and $\mathrm{CP}_{T^{(1)}}(x) = x(4 + 5x^2)(7 + 2x^2)$ up to some non-zero constants. Because

$$\mathrm{resultant}(f_1(x), \mathrm{MP}_{T^{(1)}}(x), x) \neq 0,$$

we get

$$S^{(1)} = \texttt{nearzero}(f_1(x), \mathrm{CP}_{T^{(1)}} \cdot \mathrm{MP}_{T^{(1)}}, x)$$

$$= \left[\left[\frac{-3}{8}, \frac{-5}{16}\right], \left[\frac{5}{16}, \frac{3}{8}\right]\right].$$

Obviously, the first interval need not to be considered in the following. So
$S^{(1)} = \left[\frac{5}{16}, \frac{3}{8}\right]$.

Step 2. $S^{(1)}$ has only one interval $I = \left[\frac{5}{16}, \frac{3}{8}\right]$.

Step 2a. $V_I = \left\{v_1^{(1)} = \frac{5}{16},\ v_2^{(1)} = \frac{3}{8}\right\}$.

Step 2b. Substituting $x = v_1^{(1)} = \frac{5}{16}$ into $T^{(1)}$ and deleting f_1 from it, we get

$$T_1^{(2)} : \begin{cases} f_2 = 1 + \frac{25}{16}y - 5y^2 = 0, \\ f_3 = 30z^2 - (20y + \frac{25}{4})z + \frac{25}{8}y - 11 = 0, \\ y \geq 0. \end{cases}$$

Now $\mathrm{MP}_{T_1^{(2)}}(y) = -1$ and $\mathrm{CP}_{T_1^{(2)}}(y) = (\frac{4349}{1280} - \frac{5}{16}y + y^2)y$, by

$$\texttt{nearzero}(f_2(y), \mathrm{CP}_{T_1^{(2)}} \cdot \mathrm{MP}_{T_1^{(2)}}, y),$$

we get $R_1^{(2)} = \left[\left[\frac{-3}{8}, \frac{-5}{16}\right], \left[\frac{5}{8}, \frac{11}{16}\right]\right]$. Obviously, the first interval need not to be considered in the following, so, $R_1^{(2)} = \left[\frac{5}{8}, \frac{11}{16}\right]$. Similarly, by substituting $x = v_2^{(1)} = \frac{3}{8}$ into $T^{(1)}$, we get $R_2^{(2)} = \left[\frac{5}{8}, \frac{11}{16}\right]$.

Step 2c. Merge $R_1^{(2)}$ and $R_2^{(2)}$ into $R^{(2)} : \left[\frac{5}{8}, \frac{11}{16}\right]$ and let $S_I^{(2)} = I \times R^{(2)}$.

Step 3. Because $S^{(1)}$ has only one interval, we have

$$S^{(2)} = S_I^{(2)} = \left[\frac{5}{16}, \frac{3}{8}\right] \times \left[\frac{5}{8}, \frac{11}{16}\right].$$

Now, $i = 2 < s = 3$, so, repeat Step 2 for $S^{(2)}$.

Step 2a. $S^{(2)}$ has only one element $I = \left[\frac{5}{16}, \frac{3}{8}\right] \times \left[\frac{5}{8}, \frac{11}{16}\right]$ and $V_I = \left\{(v_1^{(1)}, v_1^{(2)}) = \left(\frac{5}{16}, \frac{5}{8}\right),\ (v_2^{(1)}, v_2^{(2)}) = \left(\frac{5}{16}, \frac{11}{16}\right),\right.$ $\left.(v_3^{(1)}, v_3^{(2)}) = \left(\frac{3}{8}, \frac{5}{8}\right), (v_4^{(1)}, v_4^{(2)}) = \left(\frac{3}{8}, \frac{11}{16}\right)\right\}$.

Step 2b. Substituting $x = v_1^{(1)} = \frac{5}{16},\ y = v_1^{(2)} = \frac{5}{8}$ into $T^{(1)}$ and deleting f_1, f_2 from it, we get $T_1^{(3)} : \{f_3 = 640z^2 - 400z - 193 = 0\}$. Because this is the last equation in the ascending chain, we let $\mathrm{CP}_{T_1^{(3)}} \cdot \mathrm{MP}_{T_1^{(3)}} = 1$ and, by $\texttt{nearzero}(f_3(z), 1, z)$, get $R_1^{(3)} = [[-1, 0], [0, 1]]$. Similarly, we have $R_2^{(3)} = R_3^{(3)} = R_4^{(3)} = [[-1, 0], [0, 1]]$.

Step 2c. Merge $R_1^{(3)}, R_2^{(3)}, R_3^{(3)}$ and $R_4^{(3)}$ into $R^{(3)} : [[-1,0],[0,1]]$ and let $S_I^{(3)} = I \times R^{(3)}$.

Because $S^{(2)}$ has only one element, we have

$$S^{(3)} = S_I^{(3)} = \left[\left[\frac{5}{16}, \frac{3}{8}\right] \times \left[\frac{5}{8}, \frac{11}{16}\right] \times [-1,0],\right.$$

$$\left.\left[\frac{5}{16}, \frac{3}{8}\right] \times \left[\frac{5}{8}, \frac{11}{16}\right] \times [0,1]\right].$$

Now, $i = 3 = s$, so, go to Step 4 and output

$$\left[\left[\left[\frac{5}{16}, \frac{3}{8}\right], \left[\frac{5}{8}, \frac{11}{16}\right], [-1,0]\right], \left[\left[\frac{5}{16}, \frac{3}{8}\right], \left[\frac{5}{8}, \frac{11}{16}\right], [0,1]\right]\right].$$

2.3. *Realzero* and Examples

Our method has been implemented as a Maple program `realzero` in our package. In general, for a SAS, the computation of `realzero` consists of three main steps. First, by the Ritt-Wu method, transform the system of equations into one or more systems in triangular form. In our implementation, we use `wsolve`[33], a program which realizes Wu's method under Maple. Second, for each component, check whether it is a regular TSA and, if not, transform it into regular TSAs by Theorem 8. Third, apply `REALZERO` to each resulting regular TSA.

There are three basic kinds of calling sequences for a constant-coefficient SAS:

`realzero`$([p_1, ..., p_n], [q_1, ..., q_r], [g_1, ..., g_t], [h_1, ..., h_m], [x_1, ..., x_s]);$
`realzero`$([p_1, ..., p_n], [q_1, ..., q_r], [g_1, ..., g_t], [h_1, ..., h_m], [x_1, ..., x_s], width);$
`realzero`$([p_1, ..., p_n], [q_1, ..., q_r], [g_1, ..., g_t], [h_1, ..., h_m], [x_1, ..., x_s], [w_1, ..., w_s]);$.

The command `realzero` returns a list of isolating intervals for all real solutions of the input system or reports that the method does not work on some components. If the 6th parameter "width", a positive number, is given, the maximal size of the output intervals is less than or equal to this number. If the 6th parameter is a list of positive numbers, $[w_1, ..., w_s]$, the maximal sizes of the output intervals on $x_1, ..., x_s$ are less than or equal to $w_1, ..., w_s$, respectively. If the 6th parameter is omitted, the most convenient width is used for each interval returned. That is to say, the isolating intervals for certain x_i are returned provided that they do not intersect with each other.

Example 11: This is a problem of solving geometric constraints: Are we able to construct a triangle with elements $a = 1, R = 1$ and $h_a = \frac{1}{10}$ where a, h_a and R denote the side-length, altitude, and circumradius, respectively?

A result given by Mitrinovic *et al.*[27] says that there exists a triangle with elements a, R, h_a if and only if $R1 = 2R - a \geq 0$ and $R2 = 8Rh_a - 4h_a^2 - a^2 \geq 0$. From our study[49] (also see Section 4 in this paper for details), we know that the result is incorrect. We can also see this from the following computations. For $a = 1, R = 1, h_a = \frac{1}{10}$, we have $R1 > 0, R2 < 0$ and

$$\begin{cases} f_1 = 1/100 - 4s(s-1)(s-b)(s-c) = 0, \\ f_2 = 1/5 - bc = 0, \\ f_3 = 2s - 1 - b - c = 0, \\ b > 0, c > 0, b + c - 1 > 0, 1 + c - b > 0, 1 + b - c > 0, \end{cases}$$

where s is the half perimeter and b, c are the lengths of the other two sides, respectively. Calling

$$\texttt{realzero}\,([f_1, f_2, f_3], [\,], [b, c, b + c - 1, 1 + c - b, 1 + b - c], [\,], [s, b, c]);$$

we get

$$\left[\left[\left[\frac{259}{256}, \frac{519}{512}\right], \left[\frac{33}{128}, \frac{17}{64}\right], \left[\frac{97}{128}, \frac{197}{256}\right]\right], \left[\left[\frac{259}{256}, \frac{519}{512}\right], \left[\frac{97}{128}, \frac{99}{128}\right], \left[\frac{1}{4}, \frac{69}{256}\right]\right],\right.$$

$$\left.\left[\left[\frac{297}{256}, \frac{595}{512}\right], \left[\frac{11}{64}, \frac{23}{128}\right], \left[\frac{73}{64}, \frac{295}{256}\right]\right], \left[\left[\frac{297}{256}, \frac{595}{512}\right], \left[\frac{73}{64}, \frac{37}{32}\right], \left[\frac{21}{128}, \frac{47}{256}\right]\right]\right],$$

which means that there are two different triangles with elements $a = 1, R = 1$ and $h_a = 10^{-1}$ since b and c are symmetric in the system. The time spent for the computation on a PC (Pentium IV/2.8G) with Maple 8 is 0.2s. Furthermore, say, setting $width = 10^{-6}$ in the calling sequence:

$$\texttt{realzero}\,([f_1, f_2, f_3], [\,], [b, c, b + c - 1, 1 + c - b, 1 + b - c], [\,], [s, b, c], 10^{-6});$$

we obtain a much more accurate result,

$$\left[\left[\left[\frac{10624409}{10485760}, \frac{1062441}{1048576}\right], \left[\frac{4386135}{16777216}, \frac{4386137}{16777216}\right], \left[\frac{64173779}{83886080}, \frac{12834761}{16777216}\right]\right],\right.$$

$$\left[\left[\frac{10624409}{10485760}, \frac{1062441}{1048576}\right], \left[\frac{3208689}{4194304}, \frac{12834761}{16777216}\right], \left[\frac{21930659}{83886080}, \frac{1096535}{4194304}\right]\right],$$

$$\left[\left[\frac{152143}{131072}, \frac{12171441}{10485760}\right], \left[\frac{731239}{4194304}, \frac{1462479}{8388608}\right], \left[\frac{9623217}{8388608}, \frac{24058049}{20971520}\right]\right],$$

$$\left.\left[\left[\frac{152143}{131072}, \frac{12171441}{10485760}\right], \left[\frac{9623217}{8388608}, \frac{19246439}{16777216}\right], \left[\frac{2924953}{16777216}, \frac{7312403}{41943040}\right]\right]\right].$$

The time spent is 0.3s.

Example 12: [15] Which triangles can occur as sections of a regular tetrahedron by planes which separate one vertex from the other three? In fact, this is one of the special cases of p-3-p problem which originates from camera calibration. In Section 4, making use of another program called "DISCOVERER"[49], we have got the so-called complete solution classification of this problem.

Now, let $1, a, b$ be the lengths of the three sides of the triangle (assume $b \geq a \geq 1$), and x, y, z the distances from the vertex to the three vertexes of the triangle respectively and suppose that (a, b) is the real roots of $\{a^2 - 1 + b - b^2 = 0, 3b^6 + 56b^4 - 122b^3 + 56b^2 + 3 = 0\}$. We want to find x, y and z. Thus, the system is

$$\begin{cases} h_1 = x^2 + y^2 - xy - 1 = 0, \\ h_2 = y^2 + z^2 - yz - a^2 = 0, \\ h_3 = z^2 + x^2 - zx - b^2 = 0, \\ h_4 = a^2 - 1 + b - b^2 = 0, \\ h_5 = 3b^6 + 56b^4 - 122b^3 + 56b^2 + 3 = 0, \\ x > 0, y > 0, z > 0, a - 1 \geq 0, b - a \geq 0, a + 1 - b > 0. \end{cases}$$

Call

$\texttt{realzero}\,([h_1, h_2, h_3, h_4, h_5], [b - a, a - 1], [x, y, z, a + 1 - b], [\,], [b, a, x, y, z]);$

the output is

$$\left[\left[\left[\frac{162993}{131072}, \frac{81497}{65536}\right], \left[\frac{73}{64}, \frac{147}{128}\right], \left[\frac{1181}{1024}, \frac{2363}{2048}\right], \left[\frac{1349206836}{2188300897}, \frac{348432792}{556866289}\right],\right.\right.$$

$$\left.\left.\left[\frac{3247431090114025}{2465566125550592}, \frac{2029443732706 41}{1540423210501 12}\right]\right]\right].$$

The time spent is 15.02s. Setting $width = 10^{-6}$ in the calling sequence:

$\texttt{realzero}\,([h_1, h_2, h_3, h_4, h_5], [b-a, a-1], [x, y, z, a+1-b], [\,], [b, a, x, y, z], 10^{-6});$

we obtain a much more accurate result,

$$\left[\left[\left[\frac{162993137}{131072000}, \frac{1303945097}{1048576000}\right], \left[\frac{1225595355}{1073741824}, \frac{1225595357}{1073741824}\right], \left[\frac{77410187}{67108864}, \frac{154820375}{134217728}\right],\right.\right.$$

$$\left[\frac{5607413795199569707 1921875}{9010681213432120850 1993472}, \frac{1057264334012463994320375}{1698941787575418678673408}\right],$$

$$\left.\left.\left[\frac{352619062363191326364463801220259211}{267676127050613514331758788608000000}, \frac{557140543045141920592067747791 23}{422930119239067150975269601280 00}\right]\right]\right].$$

The time spent is 19.95s.

3. Prove or Disprove Propositions

Let Φ be a semi-algebraic system, Φ_0 a polynomial equation, inequation or inequality. Prove or disprove that $\Phi \Rightarrow \Phi_0$. Obviously, the statement is true if and only if system $\Phi \wedge \neg \Phi_0$ is inconsistent, where $\neg \Phi_0$ stands for the negative statement of Φ_0.

Automated theorem proving in real algebra and real geometry is always considered a difficult topic in the area of automated reasoning. An universal algorithm (such as methods for real quantifier elimination) would be of very high complexity (double exponential complexity in the number of variables for general case). Fortunately, the problem is easier for so-called constructive geometry. Roughly speaking, that is a class of problems where the geometric elements (points, lines and circles) are constructed step by step with rulers and compasses from the ones previously constructed.

An inequality of constructive geometry can be converted to an inequality of polynomial/radicals in independent parameter, with some inequality constraints. Let us see the following example:

Given real numbers $x, y, z, u_1, u_2, u_3, u_4, u_5, u_6$ satisfying the following 15 conditions

$$\begin{cases} (xy + yz + xz)^2 u_1^2 - x^3(y + z)(xy + xz + 4\,yz) = 0, \\ (xy + yz + xz)^2 u_2^2 - y^3(x + z)(xy + yz + 4\,xz) = 0, \\ (xy + yz + xz)^2 u_3^2 - z^3(x + y)(yz + xz + 4\,xy) = 0, \\ (x + y + z)(u_4^2 - x^2) - xyz = 0, \\ (x + y + z)(u_5^2 - y^2) - xyz = 0, \\ (x + y + z)(u_6^2 - z^2) - xyz = 0, \\ x > 0,\ y > 0,\ z > 0, \\ u_1 > 0,\ u_2 > 0,\ u_3 > 0,\ u_4 > 0,\ u_5 > 0,\ u_6 > 0, \end{cases} \tag{3}$$

prove that $u_1 + u_2 + u_3 \le u_4 + u_5 + u_6$.

Eliminating $u_1, \ldots, u_6$ from (3) by solving the 6 equations, we convert the proposition to the following inequality which appeared as a conjecture in Shan[31].

Example 13: Show that

$$\frac{\sqrt{x^3(y + z)(xy + xz + 4\,yz)}}{xy + yz + xz} + \frac{\sqrt{y^3(x + z)(xy + yz + 4\,xz)}}{xy + yz + xz} + \frac{\sqrt{z^3(x + y)(yz + xz + 4\,xy)}}{xy + yz + xz} \le$$

$$\sqrt{x^2 + \frac{xyz}{x+y+z}} + \sqrt{y^2 + \frac{xyz}{x+y+z}} + \sqrt{z^2 + \frac{xyz}{x+y+z}} \quad (4)$$

where $x > 0$, $y > 0$, $z > 0$.

This includes 3 variables but 6 radicals, while (3) includes 9 variables. A dimension-decreasing algorithm introduced by the first author can efficiently treat parametric radicals and maximize reduction of the dimensions. Based on this algorithm, a generic program called "BOTTEMA" was implemented on a PC computer. Thousands algebraic and geometric inequalities including hundreds of open problems have been proved or disproved in this way[23]. The total CPU time spent for proving 100 basic inequalities, which include some classical results such as Euler's Inequality, Finsler-Hadwiger's Inequality, and Gerretsen's Inequality, from Bottema *et al.*'s monograph[6] "*Geometric Inequalities*" on a PC (Pentium IV/2.8G) was less than 3 seconds. It can be seen later that the inequality class, to which our algorithm is applicable, is very inclusive.

In this section, we deal with a class of propositions which take the following form (though the algorithm is applicable to a more extensive class):

$$\Phi_1 \wedge \Phi_2 \wedge \cdots \wedge \Phi_s \Rightarrow \Phi_0, \quad (5)$$

where $\Phi_0, \Phi_1, \ldots, \Phi_s$ are *algebraic inequalities* (see Definition 14) in $x, y, z, \ldots$ etc., the hypothesis $\Phi_1 \wedge \Phi_2 \wedge \cdots \wedge \Phi_s$ defines either an open set (possibly, disconnected) or an open set with the whole/partial boundary.

Example 13 may be written as $(x > 0) \wedge (y > 0) \wedge (z > 0) \Rightarrow (4)$, where the hypothesis $(x > 0) \wedge (y > 0) \wedge (z > 0)$ defines an open set in the parametric space $\mathbb{R}^3$, so it belongs to the class we described. This class covers most of inequalities in Bottema *et al.*'s book[6] and Mitrinovic *et al.*'s book "*Recent Advances in Geometric Inequalities*"[27].

3.1. *Basic Definitions*

Before we sketch the so-called dimension-decreasing algorithm, some definitions should be introduced and illustrated.

Definition 14: Assume that $l(x, y, z, \ldots)$ and $r(x, y, z, \ldots)$ are continuous algebraic functions of $x, y, z, \ldots$. We call

$$l(x, y, z, \ldots) \le r(x, y, z, \ldots) \quad \text{or} \quad l(x, y, z, \ldots) < r(x, y, z, \ldots)$$

an *algebraic inequality* in $x, y, z \ldots$, and $l(x, y, z, \ldots) = r(x, y, z, \ldots)$ an *algebraic equality* in $x, y, z, \ldots$.

Definition 15: Assume that Φ is an algebraic inequality (or equality) in $x, y, z, \ldots$. $L(T)$ is called a *left polynomial* of Φ, provided that

- $L(T)$ is a polynomial in T, its coefficients are polynomials in $x, y, z, \ldots$ with rational coefficients;
- the left-hand side of Φ is a zero of $L(T)$.

The following item is unnecessary for this definition, but it helps to reduce the computational complexity in the process later.

- *Amongst all the polynomials satisfying the two items above,* $L(T)$ *is what has the lowest degree in* T*.*

According to this definition, $L(T) = T$ if the left-hand side is 0, a zero polynomial. The *right polynomial* of Φ, namely, $R(T)$, can be defined analogously.

Definition 16: Assume that Φ is an algebraic inequality (or equality) in $x, y, \ldots$ etc., $L(T)$ and $R(T)$ are the left and right polynomials of Φ, respectively. By $P(x, y, \ldots)$ denote the resultant of $L(T)$ and $R(T)$ with respect to T, and call it the *border polynomial* of Φ, and the surface defined by $P(x, y, \ldots) = 0$ the *border surface* of Φ, respectively.

The notions of left and right polynomials are needed in practice for computing the border surface more efficiently. In Example 13, we set

$$
\begin{aligned}
f_1 &= (xy + yz + xz)^2 u_1^2 - x^3(y + z)(xy + xz + 4\,yz),\\
f_2 &= (xy + yz + xz)^2 u_2^2 - y^3(x + z)(xy + yz + 4\,xz),\\
f_3 &= (xy + yz + xz)^2 u_3^2 - z^3(x + y)(yz + xz + 4\,xy),\\
f_4 &= (x + y + z)(u_4^2 - x^2) - xyz,\\
f_5 &= (x + y + z)(u_5^2 - y^2) - xyz,\\
f_6 &= (x + y + z)(u_6^2 - z^2) - xyz,
\end{aligned}
$$

then the left and right polynomials of (4) can be found by successive resultant computation:

$$
\begin{aligned}
&\mathrm{resultant}(\mathrm{resultant}(\mathrm{resultant}(u_1 + u_2 + u_3 - T, f_1, u_1), f_2, u_2), f_3, u_3),\\
&\mathrm{resultant}(\mathrm{resultant}(\mathrm{resultant}(u_4 + u_5 + u_6 - T, f_4, u_4), f_5, u_5), f_6, u_6).
\end{aligned}
$$

Removing the factors which do not involve T, we have

$$L(T) = (x\,y + x\,z + y\,z)^8\,T^8 - 4(x^4\,y^2 + 2\,x^4\,y\,z + x^4\,z^2 + 4\,x^3\,y^2\,z \\ + 4\,x^3\,y\,z^2 + x^2\,y^4 + 4\,x^2\,y^3\,z + 4\,x^2\,y\,z^3 + x^2\,z^4 + 2\,x\,y^4\,z + 4\,x\,y^3\,z^2 \\ + 4\,x\,y^2\,z^3 + 2\,x\,y\,z^4 + y^4\,z^2 + y^2\,z^4)(x\,y + x\,z + y\,z)^6\,T^6 + \cdots,$$

$$R(T) = (x + y + z)^4\,T^8 - 4(x^3 + x^2\,y + x^2\,z + x\,y^2 + 3\,x\,y\,z + x\,z^2 + y^3 + y^2\,z \\ + y\,z^2 + z^3)\,(x + y + z)^3\,T^6 + 2(16\,x\,y\,z^4 + 14\,x\,y^2\,z^3 + 14\,x\,y^3\,z^2 + 16\,x\,y^4\,z \\ + 14\,x^2\,y\,z^3 + 14\,x^2\,y^3\,z + 14\,x^3\,y\,z^2 + 14\,x^3\,y^2\,z + 16\,x^4\,y\,z + 3\,x^6 + 5\,x^4\,y^2 \\ + 5\,x^4\,z^2 + 5\,x^2\,y^4 + 5\,x^2\,z^4 + 5\,y^4\,z^2 + 5\,y^2\,z^4 + 21\,x^2\,y^2\,z^2 + 3\,y^6 + 3\,z^6 \\ + 6\,x^5\,y + 6\,x^5\,z + 4\,x^3\,y^3 + 4\,x^3\,z^3 + 6\,x\,y^5 + 6\,x\,z^5 + 6\,y^5\,z + 4\,y^3\,z^3 + 6\,y\,z^5) \\ (x + y + z)^2\,T^4 \\ - 4(x + y + z)(x^6 - x^4\,y^2 - x^4\,z^2 + 2\,x^3\,y^2\,z + 2\,x^3\,y\,z^2 - x^2\,y^4 + 2\,x^2\,y^3\,z \\ + 7\,x^2\,y^2\,z^2 + 2\,x^2\,y\,z^3 - x^2\,z^4 + 2\,x\,y^3\,z^2 + 2\,x\,y^2\,z^3 + y^6 - y^4\,z^2 - y^2\,z^4 + z^6) \\ (x^3 + 3\,x^2\,y + 3\,x^2\,z + 3\,x\,y^2 + 7\,x\,y\,z + 3\,x\,z^2 + y^3 + 3\,y^2\,z + 3\,y\,z^2 + z^3)\,T^2 \\ + (-6\,x\,y^2\,z^3 - 6\,x\,y^3\,z^2 - 6\,x^2\,y\,z^3 - 6\,x^2\,y^3\,z - 6\,x^3\,y\,z^2 - 6\,x^3\,y^2\,z + x^6 \\ - x^4\,y^2 - x^4\,z^2 - x^2\,y^4 - x^2\,z^4 - y^4\,z^2 - y^2\,z^4 - 9\,x^2\,y^2\,z^2 + y^6 + z^6 + 2\,x^5\,y \\ + 2\,x^5\,z - 4\,x^3\,y^3 - 4\,x^3\,z^3 + 2\,x\,y^5 + 2\,x\,z^5 + 2\,y^5\,z - 4\,y^3\,z^3 + 2\,y\,z^5)^2.$$

The successive resultant computation for $L(T)$ and $R(T)$ spent CPU time 0.13s and 0.03s, respectively, on a PC (Pentium IV/2.8G) with Maple 8. And then, It took us 33.05s to obtain the border polynomial of degree 100 with 2691 terms.

We may of course reform (4) to the equivalent one by transposition of terms, e.g.

$$\frac{\sqrt{x^3(y+z)(xy+xz+4\,yz)}}{xy+yz+xz} + \frac{\sqrt{y^3(x+z)(xy+yz+4\,xz)}}{xy+yz+xz} + \\ \frac{\sqrt{z^3(x+y)(yz+xz+4\,xy)}}{xy+yz+xz} - \sqrt{x^2 + \frac{xyz}{x+y+z}} - \sqrt{y^2 + \frac{xyz}{x+y+z}} \\ \le \sqrt{z^2 + \frac{xyz}{x+y+z}}. \qquad (6)$$

However, the left polynomial of (6) cannot be found on the same computer (with memory 256 Mb) by a Maple procedure as we did for (4),

```
f:=u1+u2+u3-u4-u5-T;
for i to 5 do f:=resultant(f,f.i,u.i) od;
```

this procedure did not terminate in 5 hours.

One might try to compute the border polynomial directly without employing left and right polynomials, that is, using the procedure

```
f:=u1+u2+u3-u4-u5-u6;
for i to 6 do f:=resultant(f,f.i,u.i) od;
```

but the situation is not better. The procedure did not terminate in 5 hours either.

Example 17: Given an algebraic inequality in x, y, z,

$$m_a + m_b + m_c \le 2\,s \tag{7}$$

where

$$\begin{aligned} m_a &= \frac{1}{2}\sqrt{2\,(x+y)^2 + 2\,(x+z)^2 - (y+z)^2},\\ m_b &= \frac{1}{2}\sqrt{2\,(y+z)^2 + 2\,(x+y)^2 - (x+z)^2},\\ m_c &= \frac{1}{2}\sqrt{2\,(x+z)^2 + 2\,(y+z)^2 - (x+y)^2},\\ s &= x + y + z \end{aligned}$$

with $x > 0,\ y > 0,\ z > 0$, compute the left, right and border polynomials.

Let

$$\begin{aligned} f_1 &= 4\,m_a^2 + (y+z)^2 - 2\,(x+y)^2 - 2\,(x+z)^2,\\ f_2 &= 4\,m_b^2 + (x+z)^2 - 2\,(y+z)^2 - 2\,(x+y)^2,\\ f_3 &= 4\,m_c^2 + (x+y)^2 - 2\,(x+z)^2 - 2\,(y+z)^2 \end{aligned}$$

and do successive resultant computation

$$\mathrm{resultant}(\mathrm{resultant}(\mathrm{resultant}(m_a + m_b + m_c - T, f_1, m_a), f_2, m_b), f_3, m_c),$$

we obtain a left polynomial of (7):

$$\begin{aligned} &T^8 - 6\,(x^2 + y^2 + z^2 + x\,y + y\,z + z\,x)\,T^6 + 9(x^4 + 2\,x\,y\,z^2 + y^4 + 2\,x\,z^3\\ &\quad + 2\,x^3\,y + z^4 + 3\,y^2\,z^2 + 2\,y^2\,z\,x + 2\,y^3\,z + 2\,y\,z^3 + 3\,x^2\,z^2 + 2\,x^3\,z + 2\,x^2\,y\,z\\ &\quad + 2\,x\,y^3 + 3\,x^2\,y^2)T^4 - (72\,x^4\,y\,z + 78\,x^3\,y\,z^2 + 4\,x^6 + 4\,y^6 + 4\,z^6 + 12\,x\,y^5\\ &\quad - 3\,x^4\,y^2 - 3\,x^2\,z^4 - 3\,x^2\,y^4 - 3\,y^4\,z^2 - 3\,y^2\,z^4 - 3\,x^4\,z^2 - 26\,x^3\,y^3 - 26\,x^3\,z^3\\ &\quad - 26\,y^3\,z^3 + 12\,x\,z^5 + 12\,y^5\,z + 12\,y\,z^5 + 12\,x^5\,z + 12\,x^5\,y + 84\,x^2\,y^2\,z^2\\ &\quad + 72\,x\,y\,z^4 + 72\,x\,y^4\,z + 78\,x\,y^3\,z^2 + 78\,x\,y^2\,z^3 + 78\,x^2\,y\,z^3 + 78\,x^3\,y^2\,z\\ &\quad + 78\,x^2\,y^3\,z)T^2 + 81\,x^2\,y^2\,z^2\,(x+y+z)^2. \end{aligned} \tag{8}$$

It is trivial to find a right polynomial for this inequality because the right-hand side contains no radicals. We simply take

$$T - 2\,(x + y + z). \tag{9}$$

Computing the resultant of (8) and (9) with respect to T, we have

$$\begin{aligned}(&144\,x^5\,y + 144\,x^5\,z + 780\,x^4\,y^2 + 1056\,x^4\,y\,z + 780\,x^4\,z^2 + 1288\,x^3\,y^3 \\ &+ 3048\,x^3\,y^2\,z + 3048\,x^3\,y\,z^2 + 1288\,x^3\,z^3 + 780\,x^2\,y^4 + 3048\,x^2\,y^3\,z \\ &+ 5073\,x^2\,y^2\,z^2 + 3048\,x^2\,y\,z^3 + 780\,x^2\,z^4 + 144\,x\,y^5 + 1056\,x\,y^4\,z \\ &+ 3048\,x\,y^3\,z^2 + 3048\,x\,y^2\,z^3 + 1056\,x\,y\,z^4 + 144\,x\,z^5 + 144\,y^5\,z \\ &+ 780\,y^4\,z^2 + 1288\,y^3\,z^3 + 780\,y^2\,z^4 + 144\,y\,z^5)(x + y + z)^2.\end{aligned}$$

Removing the non-vanishing factor $(x+y+z)^2$, we obtain the border surface

$$\begin{aligned}&144\,x^5\,y + 144\,x^5\,z + 780\,x^4\,y^2 + 1056\,x^4\,y\,z + 780\,x^4\,z^2 + 1288\,x^3\,y^3 \\ &+ 3048\,x^3\,y^2\,z + 3048\,x^3\,y\,z^2 + 1288\,x^3\,z^3 + 780\,x^2\,y^4 + 3048\,x^2\,y^3\,z \\ &+ 5073\,x^2\,y^2\,z^2 + 3048\,x^2\,y\,z^3 + 780\,x^2\,z^4 + 144\,x\,y^5 + 1056\,x\,y^4\,z \\ &+ 3048\,x\,y^3\,z^2 + 3048\,x\,y^2\,z^3 + 1056\,x\,y\,z^4 + 144\,x\,z^5 + 144\,y^5\,z \\ &+ 780\,y^4\,z^2 + 1288\,y^3\,z^3 + 780\,y^2\,z^4 + 144\,y\,z^5 = 0.\end{aligned} \tag{10}$$

3.2. *The Dimension-decreasing Algorithm*

We take the following procedures when the conclusion Φ_0 in (5) is of type $\leq$. (As for Φ_0 of type $<$, what we need to do in additional is to verify if the equation $l_0(x, y, \ldots) - r_0(x, y, \ldots) = 0$ has no real solutions under the hypothesis, where $l_0(x, y, \ldots)$ and $r_0(x, y, \ldots)$ denote the left- and right-hand sides of Φ_0, respectively.)

- Find the border surfaces of the inequalities $\Phi_0,\ \Phi_1,\ \ldots,\ \Phi_s$.
- These border surfaces decompose the parametric space into a finite number of cells. Among them we just take all the connected open sets, $D_1, D_2, \ldots, D_k$, and discard the lower dimensional cells. Choose at least one test point in every connected open set, say, $(x_\nu, y_\nu, \ldots) \in D_\nu,\ \nu = 0, 1, \ldots, k$. This step can be done by an incomplete cylindrical algebraic decomposition which is much easier than the complete one since all the lower dimensional cells were discarded. Furthermore, we can make every test point a rational point because it is chosen in an open set.

- We only need to check the proposition for such a finite number of test points, $(x_1, y_1, \ldots)$, …, $(x_k, y_k, \ldots)$. The statement is true if and only if it holds over these test values.

The proof of the correctness of the method is sketched as follows.

Denote the left-, right-hand sides and border surface of Φ_μ by $l_\mu(x, y, \ldots)$, $r_\mu(x, y, \ldots)$ and $P_\mu(x, y, \ldots) = 0$, respectively, and

$$\delta_\mu(x, y, \ldots) \stackrel{\text{def}}{=} l_\mu(x, y, \ldots) - r_\mu(x, y, \ldots),$$

for $\mu = 0, \ldots, s$.

The set of real zeros of all the $\delta_\mu(x, y, \ldots)$ is a closed set, so its complementary set, say Δ, is an open set. On other hand, the set

$$D \stackrel{\text{def}}{=} D_1 \cup \cdots \cup D_k$$

is exactly the complementary set of real zeros of all the $P_\mu(x, y, \ldots)$.

We have $D \subset \Delta$ since any zero of $\delta_\mu(x, y, \ldots)$ must be a zero of $P_\mu(x, y, \ldots)$. By $\Delta_1, \ldots, \Delta_t$ denote all the connected components of Δ, so each one is a connected open set. Every Δ_λ must contain a point of D for an open set cannot be filled with the real zeros of all the $P_\mu(x, y, \ldots)$. Assume that Δ_λ contains a point of D_i, some connected component of D. Then, $D_i \subset \Delta_\lambda$ because it is impossible that two different components of Δ both intersect D_i. By step 2, D_i contains a test point $(x_i, y_i, \ldots)$. So, every Δ_λ contains at least one test point obtained from step 2.

Thus, $\delta_\mu(x, y, \ldots)$ keeps the same sign over Δ_λ as that of $\delta_\mu(x_{i_\lambda}, y_{i_\lambda}, \ldots)$ where $(x_{i_\lambda}, y_{i_\lambda} \ldots)$ is a test point in Δ_λ, for $\lambda = 1, \ldots, t$; $\mu = 0, \ldots, s$. Otherwise, if there is some point $(x', y', \ldots) \in \Delta_\lambda$ that $\delta_\mu(x', y', \ldots)$ has the opposite sign to $\delta_\mu(x_{i_\lambda}, y_{i_\lambda}, \ldots)$, connecting two points $(x', y', \ldots)$ and $(x_{i_\lambda}, y_{i_\lambda}, \ldots)$ with a path Γ such that $\Gamma \subset \Delta_\lambda$, then there is a point $(\bar{x}, \bar{y}, \ldots) \in \Gamma$ such that $\delta_\mu(\bar{x}, \bar{y}, \ldots) = 0$, a contradiction!

By $A \cup B$ denote the set defined by the hypothesis, where A is an open set defined by

$$(\delta_1(x, y, \ldots) < 0) \wedge \cdots \wedge (\delta_s(x, y, \ldots) < 0),$$

that consists of a number of connected components of Δ and some real zeros of $\delta_0(x, y, \ldots)$, namely $A = Q \cup S$ where $Q = \Delta_1 \cup \cdots \cup \Delta_j$ and S is a set of some real zeros of $\delta_0(x, y, \ldots)$. And B is the whole or partial boundary of A, that consists of some real zeros of $\delta_\mu(x, y, \ldots)$ for $\mu = 1, \ldots, s$.

Now, let us verify whether $\delta_0 < 0$ holds for all the test points in A, one by one. If there is a test point whereat $\delta_0 > 0$, then the proposition is false.

Otherwise, $\delta_0 < 0$ holds over Q because every connected component of Q contains a test point and δ_0 keeps the same sign over each component Δ_λ, hence $\delta_0 \leq 0$ holds over A by continuity, so it also holds over $A \cup B$, i.e., the proposition is true.

The above procedures sometimes may be simplified. When the conclusion Φ_0 belongs to an inequality class called "class CGR", what we need to do in step 3 is to compare the greatest roots of left and right polynomials of Φ_0 over the test values.

Definition 18: An algebraic inequality is said to belong to *class CGR* if its left-hand side is the greatest (real) root of the left polynomial $L(T)$, and the right-hand side is that of the right polynomial $R(T)$.

It is obvious in Example 13 that the left- and right-hand sides of the inequality (4) are the greatest roots of $L(T)$ and $R(T)$, respectively, because all the radicals have got positive signs. Thus, the inequality belongs to class CGR. What we need to do is to verify whether the greatest root of $L(T)$ is less than or equal to that of $R(T)$, that is much easier than determine which is greater between two complicated radicals, in the sense of accurate computation.

If an inequality involves only mono-layer radicals, then it always can be transformed into an equivalent one which belongs to class CGR by transposition of terms. Actually, most of the inequalities in Bottema *et al.*[6] and Mitrinovic *et al.*[27], including most of the examples in this section, belong to the class CGR. For some more material, see Yang[47].

3.3. *Inequalities on Triangles*

An absolute majority of the hundreds inequalities discussed in Bottema *et al.*[6] are on triangles, so are the thousands appeared in various publications since then.

For geometric inequalities on a single triangle, usually the geometric invariants are used as global variables instead of Cartesian coordinates. By a, b, c denote the side-lengths, s the half perimeter, i.e. $\frac{1}{2}(a+b+c)$, and x, y, z denote $s-a, s-b, s-c$, respectively, as people used to do. In additional, by A, B, C the interior angles, S the area, R the circumradius, r the inradius, r_a, r_b, r_c the radii of escribed circles, h_a, h_b, h_c the altitudes, m_a, m_b, m_c the lengths of medians, w_a, w_b, w_c the lengths of interior angular bisectors, and so on.

People used to choose x, y, z as independent variables and others dependent. Sometimes, another choice is better for decreasing the degrees of polynomials occurred in the process.

An algebraic inequality $\Phi(x, y, z)$ can be regarded as a geometric inequality on a triangle if

- $x > 0,\ y > 0,\ z > 0$;
- the left- and right-hand sides of Φ, namely $l(x, y, z)$ and $r(x, y, z)$, both are homogeneous;
- $l(x, y, z)$ and $r(x, y, z)$ have the same degree.

The item 1 means that the sum of two edges of a triangle is greater than the third edge. The items 2 and 3 means that a similar transformation does not change the truth of the proposition. For example, (7) is such an inequality for its left- and right-hand sides, $m_a + m_b + m_c$ and $2\,s$, both are homogeneous functions (of x, y, z) with degree 1.

In addition, assume that the left- and right-hand sides of $\Phi(x, y, z)$, namely, $l(x, y, z)$ and $r(x, y, z)$, both are symmetric functions of x, y, z. It does not change the truth of the proposition to replace x, y, z in $l(x, y, z)$ and $r(x, y, z)$ with x', y', z' where $x' = \rho\, x,\ y' = \rho\, y,\ z' = \rho\, z$ and $\rho > 0$.

Clearly, the left and right polynomials of $\Phi(x', y', z')$, namely, $L(T,\, x', y', z')$ and $R(T,\, x', y', z')$, both are symmetric with respect to x', y', z', so they can be re-coded in the elementary symmetric functions of x', y', z', say,

$$H_l(T,\, \sigma_1, \sigma_2, \sigma_3) = L(T,\, x', y', z'), \quad H_r(T,\, \sigma_1, \sigma_2, \sigma_3) = R(T,\, x', y', z'),$$

where $\sigma_1 = x' + y' + z',\ \sigma_2 = x'y' + y'z' + z'x',\ \sigma_3 = x'y'z'$.

Setting $\rho = \sqrt{\frac{x+y+z}{x\,y\,z}}$, we have $x'y'z' = x' + y' + z'$, i.e., $\sigma_3 = \sigma_1$. Further, letting

$$s = \sigma_1 (= \sigma_3), \qquad p = \sigma_2 - 9,$$

we can transform $L(T,\, x', y', z')$ and $R(T,\, x', y', z')$ into polynomials in T, p, s, say, $F(T, p, s)$ and $G(T, p, s)$. Especially if F and G both have only even-degree terms in s, then they can be transformed into polynomials in T, p and q where $q = s^2 - 4\,p - 27$. Usually the degrees and the numbers of terms of the latter are much less than those of $L(T, x, y, z)$ and $R(T, x, y, z)$. We thus construct the border surface which is encoded in p, s or p, q, and do the decomposition described in last section on (p, s)-plane or (p, q)-plane instead of $\mathbb{R}^3$. This may reduce the computational complexity considerably for a large class of geometric inequalities. The following example is also taken from Bottema *et al.*[6].

Example 19: By w_a, w_b, w_c and s denote the interior angular bisectors and half the perimeter of a triangle, respectively. Prove

$$w_b w_c + w_c w_a + w_a w_b \le s^2.$$

It is well-known that

$$w_a = 2\,\frac{\sqrt{x\,(x+y)(x+z)(x+y+z)}}{2\,x+y+z},$$
$$w_b = 2\,\frac{\sqrt{y\,(x+y)(y+z)(x+y+z)}}{2\,y+x+z},$$
$$w_c = 2\,\frac{\sqrt{z\,(x+z)(y+z)(x+y+z)}}{2\,z+x+y},$$

and $s = x + y + z$. By successive resultant computation as above, we get a left polynomial which is of degree 20 and has 557 terms, while the right polynomial $T - (x + y + z)^2$ is very simple, and the border polynomial $P(x, y, z)$ is of degree 15 and has 136 terms.

However, if we encode the left and right polynomials in p, q, we get

$$\begin{aligned}
&(9\,p+2\,q+64)^4\,T^4 - 32\\
&\quad(4\,p+q+27)\,(p+8)\,(4\,p^2+p\,q+69\,p+10\,q+288)\,(9\,p+2\,q+64)^2\,T^2\\
&\quad-512\,(4\,p+q+27)^2\,(p+8)^2\,(9\,p+2\,q+64)^2\,T + 256(4\,p+q+27)^3\\
&\quad(p+8)^2\,(-1024-64\,p+39\,p^2-128\,q-12\,p\,q-4\,q^2+4\,p^3+p^2\,q)
\end{aligned}$$

and $T - 4\,p - q - 27$, respectively, hence the border polynomial

$$\begin{aligned}
Q(p,\,q) &= 5600256\,p^2\,q + 50331648\,p + 33554432\,q + 5532160\,p^3\\
&\quad+ 27246592\,p^2 + 3604480\,q^2 + 22872064\,p\,q + 499291\,p^4 + 16900\,p^5\\
&\quad+ 2480\,q^4 + 16\,q^5 + 143360\,q^3 + 1628160\,p\,q^2 + 22945\,p^4\,q\\
&\quad+ 591704\,p^3\,q + 11944\,p^3\,q^2 + 2968\,p^2\,q^3 + 242568\,p^2\,q^2 + 41312\,p\,q^3\\
&\quad+ 352\,p\,q^4
\end{aligned}$$

which is of degree 5 and has 20 terms only. The whole proving process in this way spend about 0.03s on the same machine.

3.4. *BOTTEMA and Examples*

As a prover, the whole program is written in Maple including the cell decomposition, without external packages employed.

On verifying an inequality with BOTTEMA, we only need to type in a proving command, then the machine will do everything else. If the statement is true, the computer screen will show "*The inequality holds*"; otherwise, it will show "*The inequality does not hold*" with a counter-example. There are three kinds of proving commands: *prove, xprove* and *yprove*.

prove – prove a geometric inequality on a triangle, or an equivalent algebraic inequality.

Calling Sequence:

prove(ineq);
prove(ineq, ineqs);

Parameters:

ineq − an inequality to be proven, which is encoded in the geometric invariants listed later.
ineqs − a list of inequalities as the hypothesis, which is encoded as well in the geometric invariants listed later.

Examples:

```
> read bottema;
> prove(a^2+b^2+c^2>=4*sqrt(3)*S+(b-c)^2+(c-a)^2+(a-b)^2);
```

The theorem holds

```
> prove(cos(A)>=cos(B),[a<=b]);
```

The theorem holds

xprove – prove an algebraic inequality with positive variables.

Calling Sequence:

xprove(ineq);
xprove(ineq, ineqs);

Parameters:

ineq − an algebraic inequality to be proven, with positive variables.
ineqs − a list of algebraic inequalities as the hypothesis, with positive variables.

Examples:

```
> read bottema;
> xprove(sqrt(u^2+v^2)+sqrt((1-u)^2+(1-v)^2)>=sqrt(2),
  [u<=1,v<=1]);
```

The theorem holds

```
> f:=(x+1)^(1/3)+sqrt(y-1)+x*y+1/x+1/y^2:
> xprove(f>=42496/10000,[y>1]);
```

The theorem holds

```
> xprove(f>=42497/10000,[y>1]);
```

with a counter example

$$\left[x = \frac{29}{32}, y = \frac{294117648}{294117647}\right]$$

The theorem does not hold

yprove – prove an algebraic inequality in general.

Calling Sequence:

yprove(ineq);
yprove(ineq, ineqs);

Parameters:

ineq – an algebraic inequality to be proven.
ineqs – a list of algebraic inequalities as the hypothesis.

Examples:

```
> read bottema;
> f:=x^6*y^6+6*x^6*y^5-6*x^5*y^6+15*x^6*y^4-36*x^5*y^5+15*x^4*y^6
    +20*x^6*y^3-90*x^5*y^4+90*x^4*y^5-20*x^3*y^6+15*x^6*y^2
    -120*x^5*y^3+225*x^4*y^4-120*x^3*y^5+15*x^2*y^6+6*x^6*y
    -90*x^5*y^2+300*x^4*y^3-300*x^3*y^4+90*x^2*y^5-6*x*y^6+x^6
    -36*x^5*y+225*x^4*y^2-400*x^3*y^3+225*x^2*y^4-36*x*y^5+y^6
    -6*x^5+90*x^4*y-300*x^3*y^2+300*x^2*y^3-90*x*y^4+6*y^5+15*x^4
    -120*x^3*y+225*x^2*y^2-120*x*y^3+15*y^4-20*x^3+90*x^2*y
    -90*x*y^2+20*y^3+16*x^2-36*x*y+16*y^2-6*x+6*y+1:
> yprove(f>=0);
```

The theorem holds

3.5. *More Examples*

All the examples in this subsection are computed by BOTTEMA on a PC (Pentium IV/2.8G) with Maple 8.

The following example is the well-known Janous' inequality[20] which was proposed as an open problem in 1986 and solved in 1988.

Example 20: Denote the three medians and perimeter of a triangle by m_a, m_b, m_c and $2\,s$, show that

$$\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} \geq \frac{5}{s}.$$

The left-hand side of the inequality implicitly contains three radicals. BOTTEMA automatically interprets the geometric proposition to algebraic one before proves it. The total CPU time spent for this example is 3.58s.

The next example was proposed as an open problem, E. 3146*, in the *Amer. Math. Monthly* **93**:(1986), 299.

Example 21: Denote the side-lengths and half perimeter of a triangle by a, b, c and s, respectively. Prove or disprove

$$2\,s(\sqrt{s-a}+\sqrt{s-b}+\sqrt{s-c}) \leq 3\,(\sqrt{bc(s-a)}+\sqrt{ca(s-b)}+\sqrt{ab(s-c)}).$$

The proof took us 9.91s on the same machine.

The following open problem appeared as Problem 169 in *Mathematical Communications* (in Chinese).

Example 22: Denote the radii of the escribed circles and the interior angle bisectors of a triangle by r_a, r_b, r_c and w_a, w_b, w_c , respectively. Prove or disprove

$$\sqrt[3]{r_a r_b r_c} \leq \frac{1}{3}(w_a + w_b + w_c).$$

In other words, *the geometric average of* r_a, r_b, r_c *is less than or equal to the arithmetic average of* w_a, w_b, w_c.

The right-hand side of the inequality implicitly contains 3 radicals. BOTTEMA proved this conjecture with CPU time 96.60s. One more conjecture proposed by J. Liu[31] was proven on the same machine with CPU time 52.36s. That is:

Example 23: Denote the side lengths, medians and interior-angle-bisectors of a triangle by a, b, c, m_a, m_b, m_c and w_a, w_b, w_c, respectively.

Prove or disprove

$$a\,m_a + b\,m_b + c\,m_c \le \frac{2}{\sqrt{3}}\,(w_a^2 + w_b^2 + w_c^2).$$

The following conjecture was first proposed by J. Garfunkel at *Crux Math.* in 1985, then re-proposed twice again by Mitrinovic *et al.*[27] and Kuang[22].

Example 24: Denote the three angles of a triangle by A, B, C. Prove or disprove

$$\cos\frac{B-C}{2} + \cos\frac{C-A}{2} + \cos\frac{A-B}{2} \le \frac{1}{\sqrt{3}}\,(\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} + \sin A + \sin B + \sin C\,).$$

It was proven with CPU time 21.75s.

A. Oppenheim studied the following inequality[27] in order to answer a problem proposed by P. Erdös.

Example 25: Let a, b, c and m_a, m_b, m_c be the side lengths and medians of a triangle, respectively. If $c = \min\{a,\, b,\, c\}$, then

$$2\,m_a + 2\,m_b + 2\,m_c \le 2\,a + 2\,b + (3\,\sqrt{3} - 4)\,c.$$

The hypothesis includes one more condition, $c = \min\{a,\, b,\, c\}$, so we type in

```
prove(2*ma+2*mb+2*mc<=2*a+2*b+(3*sqrt(3)-4)*c, [c<=a,c<=b]);
```

This took us 262.50s. If we type in

```
prove(2*ma+2*mb+2*mc<=2*a+2*b+(3*sqrt(3)-4)*c);
```

without the additional condition, the screen will show "*The inequality does not hold*" with a counter-example, $[a = 203,\, b = 706,\, c = 505]$.

A problem of positive semi-definite decision is originated from one of the conjectures proposed by B. Q. Liu[23]:

Example 26: Assume that $x > 0$, $y > 0$, $z > 0$. Prove

$$2187(y^4z^4(y+z)^4(2\,x+y+z)^8 + x^4z^4(x+z)^4(x+2\,y+z)^8 + x^4y^4(x+y)^4(x+y+2\,z)^8) - 256(x+y+z)^8(x+y)^4(x+z)^4(y+z)^4 \ge 0.$$

The polynomial after being expanded is of 201 terms with the largest coefficient (absolute value) 181394432. Usually it is non-trivial to decide a polynomial to be positive semi-definite or not, but this one took us CPU time 0.58s only, because of the homogeneity and symmetry which can help decrease the dimension and degree concerned.

There are two well-known geometric inequalities. One is the so-called "Euler's Inequality", $R \geq 2\,r$, another is $m_a \geq w_a$. They are often cited in illustration of various algorithms [9,42,43] for inequality proving. The following example makes a comparison between the two differences, $R-2\,r$ and m_a-w_a.

Example 27: Denote the circumradius and inradius of a triangle by R, r, and the median and the interior angle bisector on a certain side by m_a, w_a; prove

$$m_a - w_a \leq R - 2\,r.$$

It took us 2.86s.

The geometric inequalities which can be verified by the program, of course, are not limited to those on triangles. To prove the so-called "Ptolemy Inequality", we will use Cartesian coordinates instead of geometric invariants.

Example 28: Given four points A, B, C, D on a plane, Denote the distances between the points by AB, AC, AD, BC, BD, CD, respectively. Prove

$$AB \cdot CD + BC \cdot AD \geq AC \cdot BD. \tag{11}$$

Put $A = (-\frac{1}{2}, 0)$, $B = (x, y)$, $C = (\frac{1}{2}, 0)$, $D = (u, v)$, and convert (11) to

$$\sqrt{(-\frac{1}{2}-x)^2+y^2}\,\sqrt{(\frac{1}{2}-u)^2+v^2}+\sqrt{(x-\frac{1}{2})^2+y^2}\,\sqrt{(-\frac{1}{2}-u)^2+v^2}$$
$$\geq \sqrt{(x-u)^2+(y-v)^2}. \tag{12}$$

We only need to type in "`yprove(%)`" where `%` stands for inequality (12). The screen shows "`The inequality holds`" after running 3.83s.

According to our record, the CPU time spent (with Maple 8 on a Pentium IV/2.8G) and the numbers of the test points for above examples are listed as follows.

Example	13	92.44s	23 test points
Example	17	0.02s	1 test point
Example	19	0.03s	1 test point
Example	20	3.58s	12 test points
Example	21	9.91s	135 test points
Example	22	9.28s	4 test points
Example	23	52.36s	3 test points
Example	24	21.75s	121 test points
Example	25	262.50s	287 test points
Example	26	0.58s	2 test points
Example	27	2.86s	22 test points
Example	28	3.83s	48 test points

The time listed above includes that spent for all steps: finding the left, right and border polynomial, cell decomposition, and one-by-one sample point test, etc.

Remark 29: We have the following conclusions about the algorithm and the program.

- This program is applicable to any inequality-type theorem whose hypothesis and thesis all are inequalities in rational functions or radicals, but the thesis is of type "$\le$" or "$\ge$", and the hypothesis defines either an open set or an open set with the whole/partial boundary.
- It is beyond the capacity of this prover to deal with the algebraic functions other than the rational ones and radicals.
- It runs in a completely automatic mode, without human intervention.
- It is especially efficient for geometric inequalities on triangles. The input, in this case, is encoded in geometric invariants.

The program BOTTEMA can be used in global optimization to find the optimal values of polynomial/radical functions. See Yang[47] or Yang and Xia[50] for details.

4. Discover Inequality-type Theorems

In this section, we solve another problem about a parametric SAS: Give the necessary and sufficient conditions on the parameter of a parametric

SAS for the system to have a given number of distinct real solutions. Based on the idea in Section 2 and a partial cylindrical algebraic decomposition, we introduce a practical algorithm for the problem, which can discover new inequalities automatically, without requiring us to put forward any conjectures beforehand. The algorithm is complete for an extensive class of inequality-type theorems. Also this algorithm is applied to the classification of the real solutions of geometric constraint problems.

4.1. *Basic Definitions*

As discussed at the beginning of Section 2, a parametric SAS can be transformed into one or more systems in the following form

$$\begin{cases} f_1(U, x_1) = 0, \\ f_2(U, x_1, x_2) = 0, \\ \cdots\cdots \\ f_s(U, x_1, x_2, ..., x_s) = 0, \\ g_1(U, x_1, ..., x_s) \geq 0, ..., g_r(U, x_1, ..., x_s) \geq 0, \\ g_{r+1}(U, x_1, ..., x_s) > 0, ..., g_t(U, x_1, ..., x_s) > 0, \\ h_1(U, x_1, ..., x_s) \neq 0, ..., h_m(U, x_1, ..., x_s) \neq 0, \end{cases} \tag{13}$$

where $U = (x_{s+1}, ..., x_n)$ are viewed as parameter and are usually denoted by $U = (u_1, ..., u_d)$. We call a system in this form a *parametric* TSA.

All the definitions for a TSA are valid for a parametric TSA.

Definition 30: Given a parametric TSA T, let $\mathrm{BP}_{f_1} = \mathrm{CP}_{f_1}$ and

$$\mathrm{BP}_q = \mathrm{resultant}(\mathrm{CP}_q, f_1, x_1), \;\; q \in \{f_i, g_j, h_k | 1 < i \leq s, 1 \leq j \leq t, 1 \leq k \leq m\}.$$

We define $\mathrm{BP}_T(U) = \prod_{1\leq i\leq s} \mathrm{BP}_{f_i} \cdot \prod_{1\leq j\leq t} \mathrm{BP}_{g_j} \cdot \prod_{1\leq k\leq m} \mathrm{BP}_{h_k}$ and call it the *boundary polynomial* of T. It is also denoted by BP.

Then, a regular parametric TSA can be defined by $\mathrm{BP} \neq 0$. As remarked in Section 2, if a parametric TSA is regular we can omit the h_k's in it without loss of generality and each of the inequalities $g_j \geq 0$ $(1 \leq j \leq r)$ in the system can be treated as $g_j > 0$.

Definition 31: Given a polynomial with real symbolic coefficients, $f(x) = a_0x^n + a_1x^{n-1} + ... + a_n$, the following $2n \times 2n$ matrix in terms of the coefficients,

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_n & & & & \\ 0 & na_0 & (n-1)a_1 & \cdots & a_{n-1} & & & & \\ & a_0 & a_1 & \cdots & a_{n-1} & a_n & & & \\ & 0 & na_0 & \cdots & 2a_{n-2} & a_{n-1} & & & \\ & & & \cdots & \cdots & & & & \\ & & & \cdots & \cdots & & & & \\ & & & a_0 & a_1 & a_2 & \cdots & a_n \\ & & & 0 & na_0 & (n-1)a_1 & \cdots & a_{n-1} \end{bmatrix}$$

is called the *discrimination matrix* of $f(x)$, and denoted by $\mathrm{Discr}\,(f)$. Denote by d_k the determinant of the submatrix of $\mathrm{Discr}\,(f)$, formed by the first k rows and the first k columns for $k = 1, 2, ..., 2n$.

Definition 32: Let $D_0 = 1$ and $D_k = d_{2k},\ k = 1, ..., n$. We call the $(n+1)$-tuple $[D_0, D_1, D_2, ..., D_n]$ the *discriminant sequence* of $f(x)$, and denote it by $\mathrm{DiscrList}\,(f)$. Obviously, the last term D_n is $\mathrm{dis}(f, x)$.

Definition 33: We call the list

$$[\mathrm{sign}(A_0), \mathrm{sign}(A_1), \mathrm{sign}(A_2), \cdots, \mathrm{sign}(A_n)]$$

the *sign list* of a given sequence

Definition 34: Given a sign list $[s_1, s_2, \cdots, s_n]$, we construct a new list

$$[t_1, t_2, \cdots, t_n]$$

as follows: (which is called the *revised sign list*)

- If $[s_i, s_{i+1}, \cdots, s_{i+j}]$ is a section of the given list, where

  $$s_i \neq 0, s_{i+1} = \cdots = s_{i+j-1} = 0, s_{i+j} \neq 0,$$

  then, we replace the subsection

  $$[s_{i+1}, \cdots, s_{i+j-1}]$$

  by the first $j-1$ terms of $[-s_i, -s_i, s_i, s_i, -s_i, -s_i, s_i, s_i, \cdots]$, that is, let

  $$t_{i+r} = (-1)^{[(r+1)/2]} \cdot s_i, \quad r = 1, 2, \cdots, j-1.$$

- Otherwise, let $t_k = s_k$, i.e. no changes for other terms.

Theorem 35: Given a polynomial $f(x)$ with real coefficients,

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n,$$

if the number of sign changes of the revised sign list of

$$[D_0, D_1(f), D_2(f), \cdots, D_n(f)]$$

is ν, then the number of distinct pairs of conjugate imaginary roots of $f(x)$ equals ν. Furthermore, if the number of non-vanishing members of the revised sign list is l, then the number of distinct real roots of $f(x)$ equals $l - 1 - 2\nu$.

Definition 36: Given two polynomials $g(x)$ and

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n,$$

let

$$r(x) = \mathrm{rem}(f'g, f, x) = b_0x^{n-1} + b_1x^{n-2} + \cdots + b_{n-1}.$$

The following $2n \times 2n$ matrix

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_n & & & & \\ 0 & b_0 & b_1 & \cdots & b_{n-1} & & & & \\ & a_0 & a_1 & \cdots & a_{n-1} & a_n & & & \\ & 0 & b_0 & \cdots & b_{n-2} & b_{n-1} & & & \\ & & & \cdots & \cdots & & & & \\ & & & \cdots & \cdots & & & & \\ & & & a_0 & a_1 & a_2 & \cdots & a_n & \\ & & & 0 & b_0 & b_1 & \cdots & b_{n-1} & \end{bmatrix}$$

is called the *generalized discrimination matrix* of $f(x)$ with respect to $g(x)$, and denoted by $\mathrm{Discr}\,(f, g)$.

Definition 37: Given two polynomials $f(x)$ and $g(x)$. Let $D_0 = 1$ and denote by

$$D_1(f, g), D_2(f, g), \cdots, D_n(f, g)$$

the even order principal minors of $\mathrm{Discr}\,(f, g)$. We call

$$[D_0, D_1(f, g), D_2(f, g), \cdots, D_n(f, g)]$$

the *generalized discriminant sequence* of $f(x)$ with respect to $g(x)$, and denote it by $\mathrm{GDL}(f, g)$. Clearly, $\mathrm{GDL}(f, 1) = \mathrm{DiscrList}\,(f)$.

Theorem 38: Given two polynomials $f(x)$ and $g(x)$, if the number of sign changes of the revised sign list of GDL(f, g) is ν, and the number of non-vanishing members of the revised sign list is l, then

$$l - 1 - 2\nu = c(f, g_+) - c(f, g_-),$$

where

$$c(f, g_+) = \mathrm{card}(\{x \in R | f(x) = 0, g(x) > 0\}),$$

$$c(f, g_-) = \mathrm{card}(\{x \in R | f(x) = 0, g(x) < 0\}).$$

Definition 39: A normal ascending chain $\{f_1, ..., f_s\}$ is *simplicial* with respect to a polynomial g if either $\mathrm{prem}(g, f_s, ..., f_1) = 0$ or $\mathrm{res}(g, f_s, ..., f_1) \neq 0$.

Theorem 40: [54] For a triangular set $AS : \{f_1, ..., f_s\}$ and a polynomial g, there is an algorithm which can decompose AS into some normal ascending chains $AS_i : \{f_{i1}, f_{i2}, ..., f_{is}\}$ $(1 \le i \le n)$, such that every chain is simplicial with respect to g and this decomposition satisfies that $Zero(AS) = \bigcup_{1 \le i \le n} Zero(AS_i)$, where $Zero(\cdot)$ means the set of zeros of a given system.

Remark 41: We call this decomposition the RSD decomposition of AS with respect to g and the algorithm is called the RSD algorithm. The decomposition and the algorithm were called WR decomposition and WR algorithm respectively by Yang, Zhang and Hou[54]. Wang[35] proposed a similar decomposition algorithm. By Theorem 40, we always consider the triangular set $\{f_1, f_2, ..., f_s\}$ that appears in a TSA as a normal ascending chain, without loss of generality.

Definition 42: [24] Let D_k^t be the submatrix of Discr(f), formed by the first $2n - 2k$ rows, the first $2n - 2k - 1$ columns and the $(2n - 2k + t)$th column, where $0 \le k \le n - 1,\ 0 \le t \le 2k$. Let $|D_k^t| = \det(D_k^t)$. We call $|D_k^0|$ $(0 \le k \le n - 1)$ the kth *principal subresultant* of $f(x)$. Obviously, $|D_k^0| = D_{n-k}$ $(0 \le k \le n - 1)$.

Definition 43: [24] Let $Q_{n+1}(f, x) = f(x),\ Q_n(f, x) = f'(x)$, and for $k = 0, 1, ..., n-1$, $Q_k(f, x) = \sum_{t=0}^{k} |D_k^t| x^{k-t} = |D_k^0| x^k + |D_k^1| x^{k-1} + ... + |D_k^k|$. We call $\{Q_0(f, x), ..., Q_{n+1}(f, x)\}$ the *subresultant polynomial chain* of $f(x)$.

Theorem 44: [55] Suppose $\{f_1, f_2, ..., f_j\}$ is a normal ascending chain, where K is a field and $f_i \in K[x_1, ..., x_i]$, $(i = 1, 2, ..., j)$ and $f(x) =$

$a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$ is a polynomial in $K[x_1, ..., x_i][x]$, let $PD_k = \text{prem}(|D_k^0|, f_j, ..., f_1) = \text{prem}(D_{n-k}, f_j, ..., f_1)$, $(0 \le k < n)$. If for some $k_0 \ge 0$, $\text{res}(a_0, f_j, ..., f_1) \ne 0$ and $PD_0 = \cdots = PD_{k_0-1} = 0$, $\text{res}(|D_{k_0}^0|, f_j, ..., f_1) \ne 0$, then, we have $\gcd(f, f'_x) = Q_{k_0}(f, x)$ in $K[x_1, ..., x_j]/(f_1, ..., f_j)$.

Theorem 45: For an irregular parametric TSA T, there is an algorithm which can decompose T into regular systems T_i. Let all the distinct real solutions of a given system be denoted by $Rzero(\cdot)$; then this decomposition satisfies $Rzero(T) = \bigcup Rzero(T_i)$.

Proof: For T, BP= resultant$(f_1, \text{CP}, x_1) = 0$.

- If there is some CP_{h_k} such that resultant$(f_1, \text{CP}_{h_k}, x_1) = 0$, do the RSD decomposition of $\{f_1, ..., f_s\}$ with respect to h_k and, without loss of generality, suppose we get two new chains $\{A_1, ..., A_s\}$ and $\{B_1, ..., B_s\}$, in which $\text{prem}(h_k, A_s, ..., A_1) = 0$ but $\text{res}(h_k, B_s..., B_1) \ne 0$. If we replace $\{f_1, ..., f_s\}$ by $\{B_1, ..., B_s\}$ in T, the new system is regular and has the same real solutions as those of the original system. Obviously, another system obtained by replacing $\{f_1, ..., f_s\}$ with $\{A_1, ..., A_s\}$ in T, has no real solutions.
- If there is some CP_{g_j} such that resultant$(f_1, \text{CP}_{g_j}, x_1) = 0$, do the RSD decomposition of $\{f_1, ..., f_s\}$ with respect to g_j and suppose we get $\{A_1, ..., A_s\}$ and $\{B_1, ..., B_s\}$, in which $\text{prem}(g_j, A_s, ..., A_1) = 0$ but $\text{res}(g_j, B_s..., B_1) \ne 0$. Now, if $g_j > 0$ in T, we simply replace $\{f_1, ..., f_s\}$ by $\{B_1, ..., B_s\}$. The new system is regular and has the same real solutions as those of the original system. If $g_j \ge 0$ in T, we first get a new system T_1 by replacing $\{f_1, ..., f_s\}$ with $\{B_1, ..., B_s\}$ and then, get another new system T_2 by replacing $\{f_1, ..., f_s\}$ with $\{A_1, ..., A_s\}$ and deleting g_j from it. These two systems are both regular and we have $Rzero(T) = Rzero(T_1) \bigcup Rzero(T_2)$.
- If there is some CP_{f_i} such that resultant$(f_1, \text{CP}_{f_i}, x_1) = 0$, let $[D_1, ..., D_{n_i}]$ be the discriminant sequence of f_i with respect to x_i. First of all, we do the RSD decomposition of $\{f_1, ..., f_{i-1}\}$ with respect to D_{n_i} and suppose we get $\{A_1, ..., A_{i-1}\}$ and $\{B_1, ..., B_{i-1}\}$, in which $\text{prem}(f_i, A_{i-1}, ..., A_1) = 0$ but $\text{res}(f_i, B_{i-1}..., B_1) \ne 0$. Step 1, replacing $\{f_1, ..., f_{i-1}\}$ with $\{B_1, ..., B_{i-1}\}$, we will get a regular system. Step 2, let us consider the system obtained by replacing $\{f_1, ..., f_{i-1}\}$ with $\{A_1, ..., A_{i-1}\}$ which is still irregular. Consider D_{n_i-1}, the next term in $[D_1, ..., D_{n_i}]$.

If $\mathrm{res}(D_{n_i-1}, A_{i-1}, ..., A_1) = 0$, do the RSD decomposition of $\{A_1, ..., A_{i-1}\}$ with respect to D_{n_i-1}. Keep repeating the same procedure until at a certain step we have, for certain D_{i_0} and $\{\bar{A}_1, ..., \bar{A}_{i-1}\}$, $\mathrm{res}(D_{i_0}, \bar{A}_{i-1}, ..., \bar{A}_1) \neq 0$ and $\forall j$ $(i_0 < j \leq n_i)$, $\mathrm{prem}(D_j, \bar{A}_{i-1}, ..., \bar{A}_1) = 0$. Note that this procedure must terminate because $\{f_1, ..., f_s\}$ being a normal ascending chain implies $\mathrm{res}(I_i, f_{i-1}, ..., f_1) \neq 0$ and $D_1 = n_i I_i^2$ implies $\mathrm{res}(D_1, f_{i-1}, ..., f_1) \neq 0$. By Theorem 2.3, $\gcd(f_i, f_i') = Q_{n_i-i_0}(f_i, x_i)$ in $K[x_1, ..., x_{i-1}]/(\bar{A}_1, ..., \bar{A}_{i-1})$. Now, let $\bar{f}_i$ be the pseudo-quotient of f_i divided by $\gcd(f_i, f_i')$ and replace $\{f_1, ..., f_{i-1}, f_i\}$ with $\{\bar{A}_1, ..., \bar{A}_{i-1}, \bar{f}_i\}$, the new system will be regular. If the new regular systems are T_j $(1 \leq j \leq j_i)$, it is easy to see that $Rzero(T) = \bigcup_{1\leq j\leq j_i} Rzero(T_j)$. □

By Theorem 45, every parametric TSA in the rest of this section can be treated as a regular one.

4.2. *The Algorithm*

Let

$$ps = \{p_i | 1 \leq j \leq n\}$$

be a nonempty, finite set of polynomials. We define

$$\mathrm{mset}(ps) = \{1\} \cup \{p_{i_1} p_{i_2} \cdots p_{i_k} | 1 \leq k \leq n, 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}.$$

Given a parametric TSA T, we define

$$\begin{aligned}
&P_{s+1} = \{g_1, g_2, \cdots, g_t\};\\
&U_i = \bigcup_{q \in \mathrm{mset}(P_{i+1})} \mathrm{GDL}(f_i, q),\\
&P_i = \{h(u, x_1, \cdots, x_{i-1}) | h \in U_i\}, \quad \text{for } i = s, s-1, \cdots, 2;\\
&P_1(g_1, g_2, \cdots, g_t) = \{h(u) | h \in U_1\},
\end{aligned}$$

where U_i means the set consisting of all the polynomials in each $\mathrm{GDL}(f_i, q)$ where q belongs to $\mathrm{mset}(P_{i+1})$. Analogously, we can define $P_1(g_1, \cdots, g_j)\,(1 \leq j \leq t)$. It is clear that all the factors of the boundary polynomial, BP, of T are included in $P_1(g_1, g_2, \cdots, g_t)$. With a little abuse of notations, we write $\mathrm{BP} \subseteq P_1(g_1, g_2, \cdots, g_t)$.

Theorem 46: The necessary and sufficient condition for a parametric TSA T to have a given number of distinct real solution(s) can be expressed in terms of the signs of the polynomials in $P_1(g_1, g_2, \cdots, g_t)$.

Proof: First of all, we regard f_s and every g_i as polynomials in x_s. By Theorems 35 and 38 we know that under constraints $\{g_i \geq 0 | 1 \leq i \leq t\}$, the number of distinct real solutions of $f_s = 0$ can be determined by the signs of polynomials in P_s. Let $h_j (1 \leq j \leq l)$ be the polynomials in P_s, then we regard every h_j and f_{s-1} as polynomials in x_{s-1}. Repeating the same argument as that for f_s and g_i's, we get that, under constraints $\{g_i \geq 0 | 1 \leq i \leq t\}$, the number of distinct real solutions of $f_s = 0, f_{s-1} = 0$ can be determined by the signs of polynomials in P_{s-1}. Continuing in this way until $P_1(g_1, g_2, \cdots, g_t)$ is employed, we have that the theorem holds because the conditions obtained in each step are necessary and sufficient. □

Remark 47: Ben-Or *et al.*[5] gave a different way to define a smaller set of polynomials in the parameter for a parametric TSA which can determines the sign assignments to the g_j at roots of $\{f_1, ..., f_s\}$.

Now, theoretically speaking, we can obtain the necessary and sufficient condition for a parametric TSA T to have (exactly N distinct) real solution(s) as follows:

Step 1 Compute $P_1(g_1, g_2, \cdots, g_t)$, the set of polynomials in parameter, for T.

Step 2 By the algorithm of PCAD[10,7], we can obtain P_1-*invariant cad* D of parameter space $\mathbb{R}^d$ and its *cylindrical algebraic sample* (cas) S[40]. Roughly speaking, D is a finite set of cells such that each polynomial of P_1 keeps its sign in each cell; and S is a finite set of points obtained by taking from each cell one point at least, which is called the sample point of the cell.

Step 3 For each cell c in D and its sample point $s_c \in S$, substitute s_c into T and denote it by $T(s_c)$. Compute the number of distinct real solutions of system $T(s_c)$, in which polynomials all have constant coefficients now. At the same time, compute the signs of polynomials in $P_1(g_1, g_2, \cdots, g_t)$ on this cell by substituting s_c into them respectively. Record the signs of polynomials in $P_1(g_1, g_2, \cdots, g_t)$ when the number of distinct real solutions of system $T(s_c)$ equals to the required number N (or when the number > 0, if we are asked to find the condition for T to have real solutions). Obviously, the signs of polynomials in $P_1(g_1, g_2, \cdots, g_t)$ on cell c form a first order formula, denoted by Φ_c.

Step 4 If, in step 3, all we have recorded are $\Phi_{c_1}, \cdots, \Phi_{c_k}$, then $\Phi = \Phi_{c_1} \vee \cdots \vee \Phi_{c_k}$ is what we want.

The above algorithm is not practical in many cases since $P_1(g_1, \cdots, g_t)$ usually has too many polynomials and a complete cylindrical algebraic decomposition is usually inefficient. So, in order to make our algorithm practical, we take the following strategies. First, we give an effective algorithm to choose those polynomials which are necessary for expressing the condition from $P_1(g_1, \cdots, g_t)$. Second, we always omit the "boundaries" when use PCAD and the incompleteness caused by this omission will be fixed up later.

Theorem 48: Let a parametric TSA T be given. If $PolySet$ is a finite set of polynomials in parameter U, e.g.

$$PolySet = \{q_i(U) \in \mathbb{Z}[u_1, \cdots, u_d] | 1 \le i \le k\},$$

then by the algorithm of PCAD we can get a $PolySet$-invariant cad D of parameter space $\mathbb{R}^d$ and its cas. If $PolySet$ satisfies that

(1) the number of distinct real solutions of system T is invariant in the same cell and
(2) the numbers of distinct real solutions of system T in two distinct cells C_1 and C_2 are the same if $PolySet$ has the same sign in C_1 and C_2,

then the necessary and sufficient conditions for T to have exactly N distinct real solution(s) can be expressed by the signs of the polynomials in $PolySet$. If $PolySet$ satisfies item (1) only, then some necessary conditions for T to have exactly N distinct real solution(s) can be expressed by the signs of the polynomials in $PolySet$.

Proof: We replace parameter U in T with each sample point respectively. Because D is $PolySet$-invariant and $PolySet$ satisfies item 1, we can record the signs of polynomials in $PolySet$ and the number of distinct real solutions of T on each cell respectively. Choose all those cells on which T has N distinct real solution(s). The signs of polynomials in $PolySet$ on those cells form a first order formula, say,

$$\Phi = \Phi_1 \vee \Phi_2 \vee \cdots \vee \Phi_l,$$

where each Φ_i represents the signs of polynomials in $PolySet$ on a certain cell on which T has N distinct real solution(s). We show that Φ is the condition we want.

Given a parameter $a = (a_1, \cdots, a_d)$, if $T(a)$ has N distinct real solution(s), then a must belong to a cell on which T has N distinct real solution(s), i.e. a must satisfy a certain formula Φ_i. On the contrary, if a

satisfies a certain formula Φ_i, because T has N distinct real solution(s) on the cell represented by Φ_i and $PolySet$ satisfies item (2), we thus know that T must have N distinct real solution(s) on the cell which a belongs to. □

Theorem 49: Given a regular parametric TSA T, i.e., BP $\neq 0$. If we only consider those cells which are homeomorphic to $\mathbb{R}^d$ and do not consider those cells which are homeomorphic to $\mathbb{R}^k$ $(k < d)$ when use PCAD, then BP satisfies item 1 in Theorem 48, so a necessary condition (if we omit the parameter on those cells homeomorphic to $\mathbb{R}^k$ $(k < d)$) for system T to have N distinct real solution(s) can be expressed by the sign of BP or the signs of the factors of BP.

Proof: By PCAD, we can get a BP-invariant cad of $\mathbb{R}^d$ and its cas. Because we only consider those cells which are homeomorphic to $\mathbb{R}^d$, the signs of each BP_{f_i} and BP_{g_j} on a given cell C are invariant and do not equal 0.

First of all, by the definition of BP_{f_1}, the sign of BP_{f_1} on C is invariant implies that the number of real solutions of $f_1(U, x_1)$ is invariant on C. We regard $f_2(U, x_1, x_2)$ as a polynomial in x_2, because on C,

$$f_1(U, x_1) = 0 \quad \text{and} \quad \mathrm{BP}_{f_2} = \mathrm{res}(\mathrm{dis}(f_2, x_2), f_1, x_1) \neq 0,$$

$\mathrm{dis}(f_2, x_2) \neq 0$ on C. Thus, if we replace x_1 in f_2 with the roots of f_1, the number of real solutions of f_2 is invariant. That is to say, the signs of BP_{f_1} and BP_{f_2} being invariant on C implies the number of real solutions of $f_1 = 0, f_2 = 0$ is invariant on C; now, it's easy to see that the signs of $\mathrm{BP}_{f_1}, \cdots, \mathrm{BP}_{f_s}$ being invariant on C implies the number of real solutions of $f_1 = 0, \cdots, f_s = 0$ is invariant on C.

Secondly, by the definition of BP_{g_j}, $\mathrm{BP}_{g_j} \neq 0$ implies that the sign of g_j is invariant on C if we replace $x_1, \cdots, x_s$ in g_j with the roots of $f_1 = 0, \cdots, f_s = 0$. That completes the proof. □

By Theorem 49, for a regular parametric TSA T, we can start our algorithm from BP as follows:

Algorithm: `tofind`
Input: a regular parametric TSA T and an integer N;
Output: the necessary and sufficient condition on the parameter for T to have exactly N distinct real solution(s) provided that the parameter are not on some "boundaries".
Step 1 Let $PolySet = \mathrm{BP}$, $i = 1$.

Step 2 By the algorithm of PCAD, compute a *PolySet*-invariant cad D of the parameter space $\mathbb{R}^d$ and its cylindrical algebraic sample (cas) S. In this step, we only consider those cells homeomorphic to $\mathbb{R}^d$ and do not consider those homeomorphic to $\mathbb{R}^k$ $(k < d)$, i.e., all those cells in D are homeomorphic to $\mathbb{R}^d$ and all sample points in S are taken from cells in D.

Step 3 For each cell c in D and its sample point $s_c \in S$, substitute s_c into T and denote it by $T(s_c)$. Compute the number of distinct real solutions of system $T(s_c)$, in which polynomials all have constant coefficients now. At the same time, compute the signs of polynomials in *PolySet* on this cell by substituting s_c into them respectively. Obviously, the signs of polynomials in *PolySet* on cell c form a first order formula, denoted by Φ_c. When all the $T(s_c)$'s are computed, let

$$set_1 = \{\Phi_c |\ T \text{ has } N \text{ distinct real solution(s) on } c\},$$

$$set_0 = \{\Phi_c |\ T \text{ does not have } N \text{ distinct real solution(s) on } c\}.$$

Step 4 Decide whether all the recorded Φ_c's can form a necessary and sufficient condition or not by verifying whether $set_1 \cap set_0$ is empty or not (because of Theorems 48 and 49). If $set_1 \cap set_0 = \emptyset$, go to Step 5; If $set_1 \cap set_0 \neq \emptyset$, let

$$PolySet = PolySet \cup P_1(g_1, \cdots, g_i), \ \ i = i + 1,$$

and back to Step 2.

Step 5 If $set_1 = \{\Phi_{c_1}, \cdots, \Phi_{c_m}\}$, then $\Phi = \Phi_{c_1} \vee \cdots \vee \Phi_{c_m}$ is what we want.

Remark 50: The termination of this algorithm is guaranteed by Theorem 46.

Remark 51: In order to make our algorithm practical, we do not consider the “boundaries” when use PCAD. So, the condition obtained by this algorithm is a necessary and sufficient one if we omit the situation on the “boundaries”.

Actually, in many cases, the condition obtained by `tofind` is satisfactory enough because we do not lose too much information though it is not a necessary and sufficient one. In the following, we give a complementary algorithm which deals with the situation when parameter are on “boundaries” and thus makes the practical algorithm to be a complete one.

Given a parametric TSA T. Let $R(u_1, ..., u_d)$ be one of the polynomials in parameter to express the condition for T to have N distinct real solution(s), which are obtained by `tofind`. Now, the condition for T to have N distinct real solution(s) when parameter are on $R = 0$ is needed. We take the following steps:

Algorithm: `Tofind`

Input: a regular parametric TSA T, a boundary $R = 0$ and an integer N;

Output: the necessary and sufficient condition for T to have exactly N distinct real solution(s) when the parameter are on $R = 0$.

Step 1 Let TR be the new system by adding $R = 0$ into T. Now, we regard (u_1, X) as variables and $(u_2, ..., u_d)$ parameter, where $X = (x_1, ..., x_s)$. Then, TR is of the same type as T. If TR is not regular, by Theorem 45, we can decompose it into regular ones. So, for concision, we regard TR as a regular system.

Step 2 Let $PolySet = \mathrm{BP}_{TR}$, $i = 1$.

Step 3 By the algorithm of PCAD, compute a $PolySet$-invariant cad D of parameter space $\mathbb{R}^{d-1}$ and its cylindrical algebraic sample (cas) S.

Step 4 Let $S' = \{\}$. For every sample point $s_c \in S$, substitute s_c into $R = 0$. If the distinct real solutions of $R(s_c) = 0$ are $a_1 < ... < a_k$, then put every (a_i, s_c) $(1 \le i \le k)$ into S'.

Step 5 For every sample point $(a_j, s_c) \in S'$, substitute it into T and the new system is denoted by $T(a_j, s_c)$. Compute the number of distinct real solutions of system $T(a_j, s_c)$. At the same time, compute the signs of polynomials in $PolySet$ at s_c. Obviously, the signs of polynomials in $PolySet$ at s_c form a first order formula, denoted by Φ_c. For (a_j, s_c), we replace Φ_c by (Φ_c, j). Then, let

$$set_1 = \{(\Phi_c, j)|\ T \text{ has required real solution(s) at } (a_j, s_c)\},$$

$$set_0 = \{(\Phi_c, j)|\ T \text{ does not have required real solution(s) at } (a_j, s_c)\}.$$

Step 6 Decide whether set_1 can form a necessary and sufficient condition or not by verifying whether $set_1 \cap set_0$ is empty or not. If $set_1 \cap set_0 = \emptyset$, go to Step 7; If $set_1 \cap set_0 \neq \emptyset$, let

$$PolySet = PolySet \cup P_1(g_1, \cdots, g_i),\ i = i + 1,$$

and back to Step 3, where $P_1(g_1, \cdots, g_i)$ is defined w.r.t. TR.

Step 7 If $set_1 = \{(\Phi_{c_1}, j_1), \cdots, (\Phi_{c_m}, j_m)\}$, then $\Phi = (\Phi_{c_1}, j_1) \vee \cdots \vee (\Phi_{c_m}, j_m)$ is what we want, where (Φ_{c_i}, j_i) means the parameter

$(u_1, ..., u_d)$ should satisfy Φ_{c_i} and u_1 is the j_ith real root of $R = 0$ when $(u_2, ..., u_d)$ is fixed.

Remark 52: In Step 3 of `Tofind`, as in `tofind`, we only consider those cells homeomorphic to $\mathbb{R}^{d-1}$ and do not consider those homeomorphic to $\mathbb{R}^k$ $(k < d-1)$. Therefore, if $S(u_2, ..., u_d)$ is a member of the final *PolySet* and further result when parameter are on both $R = 0$ and $S = 0$ is needed, we just put $S = 0$ into TR and apply above algorithm again.

4.3. *DISCOVERER and Examples*

The algorithms in last subsection have been implemented as a Maple program "DISCOVERER" in our package. There are two main functions, `tofind` and `Tofind`, in DISCOVERER. They are applicable to those problems which can be formulated into a parametric SAS. Usually, we call `tofind` first to find a satisfactory condition (see Remark 51) and then, if necessary, call `Tofind` to find further results when parameter are on some boundaries.

The calling sequence in DISCOVERER for a parametric SAS T is:

$$\text{tofind}\,([p_1, \cdots, p_s], [g_1, \cdots, g_r], [g_{r+1}, \cdots, g_t], [h_1, \cdots, h_m],\\ [x_1, \cdots, x_s], [u_1, \cdots, u_d], \alpha);$$

where α has following three kind of choices:

- a non-negative integer b which means the condition for T to have exactly b distinct real solution(s);
- a range $b..c$ (b, c are non-negative integers, $b < c$) which means the condition for T to have b or $b + 1$ or $\cdots$ or c distinct real solutions;
- a range $b..w$ (b is a non-negative integer, w a name) which means the condition for T to have more than or equal to b distinct real solutions.

Similarly, the calling sequence of `Tofind` for T and some "boundaries" $R_1 = 0, ..., R_l = 0$ is:

$$\text{Tofind}\,([p_1, \cdots, p_s, R_1, \cdots, R_l], [g_1, \cdots, g_r], [g_{r+1}, \cdots, g_t],\\ [h_1, \cdots, h_m], [x_1, \cdots, x_s], [u_1, \cdots, u_d], \alpha);$$

where each R_i is a "boundary" which can be a polynomial in parameter obtained by `tofind` or a constraint polynomial in parameter.

Example 53: [15] Which triangles can occur as sections of a regular tetrahedron by planes which separate one vertex from the other three?

If we let $1, a, b$ (assume $b \geq a \geq 1$) be the lengths of three sides of the triangle, and x, y, z the distances from the vertex to the three vertexes of the triangle respectively, then, what we need is to find the necessary and sufficient condition that a, b should satisfy for the following system to have real solution(s),

$$\begin{cases} h_1 = x^2 + y^2 - xy - 1 = 0, \\ h_2 = y^2 + z^2 - yz - a^2 = 0, \\ h_3 = z^2 + x^2 - zx - b^2 = 0, \\ x > 0, y > 0, z > 0, a - 1 \geq 0, b - a \geq 0, a + 1 - b > 0. \end{cases}$$

With our program DISCOVERER, we attack this problem by following two steps. First of all, we type in:

$$\text{tofind}\,([h_1, h_2, h_3], [a-1, b-a], [x, y, z, a+1-b], [\ ], [x, y, z], [a, b], 1..n);$$

DISCOVERER runs 3 seconds on a PC (Pentium IV/2.8G) with Maple 8, and outputs

FINAL RESULT :

The system has required real solution(s) IF AND ONLY IF

$$[0 < R1, 0 < R2]$$

or

$$[0 < R1, R2 < 0, 0 < R3]$$

where

$$R1 = a^2 + a + 1 - b^2$$

$$R2 = a^2 - 1 + b - b^2$$

$$\begin{aligned} R3 = 1 &- \frac{8}{3}a^2 - \frac{8}{3}b^2 + \frac{16}{9}a^8 - \frac{68}{27}b^6a^2 + \frac{241}{81}b^4a^4 - \frac{68}{27}b^2a^6 \\ &- \frac{68}{27}b^4a^2 - \frac{68}{27}b^2a^4 - \frac{2}{9}b^6 + \frac{16}{9}b^8 - \frac{2}{9}a^6 + \frac{46}{9}b^2a^2 \\ &+ \frac{16}{9}b^4 + \frac{16}{9}a^4 + \frac{46}{9}b^2a^8 + \frac{46}{9}b^8a^2 - \frac{68}{27}b^6a^4 - \frac{68}{27}b^4a^6 \\ &+ \frac{16}{9}b^4a^8 - \frac{8}{3}b^{10}a^2 + \frac{16}{9}b^8a^4 - \frac{2}{9}b^6a^6 - \frac{8}{3}b^2a^{10} - \frac{8}{3}b^{10} \\ &+ b^{12} - \frac{8}{3}a^{10} + a^{12} \end{aligned}$$

PROVIDED THAT :

$$-b+a \neq 0$$
$$a-1 \neq 0$$
$$b-1 \neq 0$$
$$a^2-1+b-b^2 \neq 0$$
$$a^2-1-b-b^2 \neq 0$$
$$a^2-a+1-b^2 \neq 0$$
$$a^2+a+1-b^2 \neq 0$$
$$a^2-1-ab+b^2 \neq 0$$
$$a^2-1+ab+b^2 \neq 0$$
$$R3 \neq 0$$

Folke[15] gave a sufficient condition that any triangle with two angles $> 60°$ is a possible section. It is easy to see that this condition is equivalent to $[R1 > 0,\ R2 > 0\,]$.

Now, if parameter a, b are not on the boundaries (that is, $R1 = 0$, $R2 = 0$, $R3 = 0$, $a-1 = 0$, $b-a = 0, ...$), the condition obtained above is already a necessary and sufficient one. But, strictly speaking, to get a necessary and sufficient condition, we have to give the result when a, b are on the boundaries. Thus, we take the second step. If we want to know the result when a, b are on a certain boundary, say $R2$, we only need to type in

Tofind $([h_1, h_2, h_3, R2], [a-1, b-a], [x, y, z, a+1-b], [\,], [x, y, z], [a, b], 1..n)$;

DISCOVERER outputs that (0.44 seconds)

FINAL RESULT:

The system has required real solution(s) IF AND ONLY IF

$$[S1 < 0, (2)R2]$$

where

$$S1 = b^6 + \frac{56}{3}b^4 - \frac{122}{3}b^3 + \frac{56}{3}b^2 + 1$$

PROVIDED THAT :

$$b-1 \neq 0$$
$$S1 \neq 0$$

$[S1 < 0, (2)R2]$ in the output means a point (a_0, b_0) in the parametric plane should satisfy that $S1 < 0$ and a_0 is the second root (from the smallest one up) of $R2(a, b_0) = 0$. Furthermore, the situation when (a, b) is on

$R2 = 0 \wedge b - 1 = 0$ or $R2 = 0 \wedge S1 = 0$ can be determined by typing in respectively:

Tofind $([h_1, h_2, h_3, R2, b{-}1], [a{-}1, b{-}a], [x, y, z, a{+}1{-}b], [\,], [x, y, z], [b, a], 1..n)$;

Tofind $([h_1, h_2, h_3, R2, S1], [a{-}1, b{-}a], [x, y, z, a{+}1{-}b], [\,], [x, y, z], [b, a], 1..n)$;

The outputs both are:

The system has 1 real solution!

The timings of the computations are 1.13 and 1.44 seconds, respectively.

By this way together with some interactive computations, we finally get the condition for the system to have real solution(s):

$$[0 < R1, 0 < R2, R3 \leq 0, 0 < a - 1, 0 \leq b - a, 0 < a + 1 - b]$$
$$\text{or}$$
$$[0 < R1, 0 \leq R3, 0 \leq a - 1, 0 \leq b - a, 0 < a + 1 - b].$$

Actually, by our algorithm and program, we can do more than the request to this problem. If we type in respectively

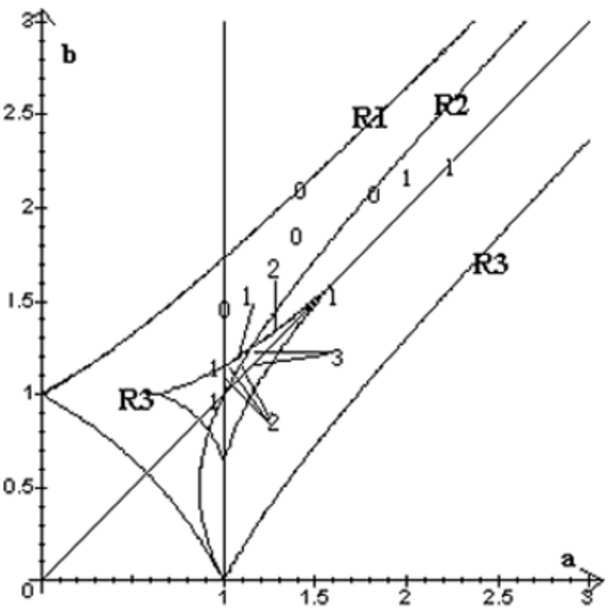

Fig. 1. The complete solution classification of Example 53.

tofind $([h_1, h_2, h_3], [a-1, b-a], [x, y, z, a+1-b], [\ ], [x, y, z], [a, b], 1)$;

tofind $([h_1, h_2, h_3], [a-1, b-a], [x, y, z, a+1-b], [\ ], [x, y, z], [a, b], 2)$;

tofind $([h_1, h_2, h_3], [a-1, b-a], [x, y, z, a+1-b], [\ ], [x, y, z], [a, b], 3)$;

we will get the condition for the above system to have exactly 1 or 2 or 3 real solution(s) respectively. By this way, we obtain the so-called complete solution classification of this problem, as indicated in Fig. 1. The number $(0, 1, 2$ or $3)$ in a certain region indicates the number of distinct real solutions of the system when the parameter a, b are on the region.

Example 54: It is well-known that for a triangle there are four tritangent circles (i.e. one inscribed circle and three escribed circles) and a Feuerbach circle (i.e. nine-point-circle) whose radius equals half the circumradius. Given a triangle ABC whose vertices $B(1, 0)$ and $C(-1, 0)$ are fixed and the vertex $A(u_1, u_2)$ depends on two parameters, we want to find the conditions on u_1, u_2 such that there are four, three, two, one or none of the tritangent circles whose radius are smaller than that of Feuerbach circle, respectively.

By a routine computation, the system to be dealt with is

$$\begin{cases} f = 16x^2u_2^2 - (u_1^2 + 2u_1 + 1 + u_2^2)(1 - 2u_1 + u_1^2 + u_2^2) = 0, \\ i = y^4u_2 + (2 - 2u_2^2 - 2u_1^2)y^3 + u_2(u_1^2 - 5 + u_2^2)y^2 + 4u_2^2y - u_2^3 = 0, \\ x > 0,\ x^2 - y^2 > 0, \end{cases}$$

where x is the radius of the Feuerbach circle and $|y|$ are the radii of the four tritangent circles.

We type in

$$\begin{array}{l} \text{tofind}([f, i], [\ ], [x, x^2 - y^2], [\ ], [x, y], [u_1, u_2], 4); \\ \text{tofind}([f, i], [\ ], [x, x^2 - y^2], [\ ], [x, y], [u_1, u_2], 3); \\ \text{tofind}([f, i], [\ ], [x, x^2 - y^2], [\ ], [x, y], [u_1, u_2], 2); \\ \text{tofind}([f, i], [\ ], [x, x^2 - y^2], [\ ], [x, y], [u_1, u_2], 1); \\ \text{tofind}([f, i], [\ ], [x, x^2 - y^2], [\ ], [x, y], [u_1, u_2], 0); \end{array}$$

respectively and get the following results (for concision, we rearrange the outputs in a simpler form):

FINAL RESULT :

The system has 3 (distinct) real solutions IF AND ONLY IF

$$[R1 < 0, R2 > 0, R3 < 0]$$

The system has 2 (distinct) real solutions IF AND ONLY IF

$$[R1 > 0]$$

The system has 1 (distinct) real solution IF AND ONLY IF

$$[R1 < 0, R2 < 0]$$

or

$$[R1 < 0, R2 > 0, R3 > 0]$$

The system does not have 0 or 4 real solution(s).

where

$$\begin{aligned} R1 = &-7 + 20u_2^6u_1^2 + 20u_2^2 + 28u_1^2 - 52u_1^2u_2^2 - 42u_1^4 + 70u_2^4 - 204u_2^6 \\ &+ 68u_2^4u_1^2 + 9u_2^8 + 6u_2^4u_1^4 + 28u_1^6 - 7u_1^8 + 44u_1^4u_2^2 - 12u_2^2u_1^6, \end{aligned}$$

$$\begin{aligned} R2 = &189 + 189u_1^{12} + 720u_2^2 - 1134u_1^2 - 1977u_2^8 + 2835u_1^4 - 1235u_2^4 \\ &- 3560u_2^6 - 3780u_1^6 + 2835u_1^8 - 8088u_2^6u_1^2 - 1968u_1^2u_2^2 + 2332u_2^4u_1^2 \\ &+ 558u_2^4u_1^4 + 672u_1^4u_2^2 + 2592u_2^2u_1^6 + 984u_2^6u_1^6 - 1566u_2^8u_1^2 - 40u_2^{10}u_1^2 \\ &+ 135u_2^8u_1^4 - 2776u_2^6u_1^4 - 3172u_2^4u_1^6 - 2928u_1^8u_2^2 + 1517u_1^8u_2^4 \\ &+ 912u_2^2u_1^{10} + 15u_2^{12} - 168u_2^{10} - 1134u_1^{10}, \end{aligned}$$

$$\begin{aligned} R3 = &-63 + 225u_2^{14}u_1^2 - 63u_1^{16} + 4284u_1^{12} - 345u_2^2 - 504u_1^2 + 515u_2^8 \\ &+ 4284u_1^4 + 485u_2^4 + 3347u_2^6 - 11592u_1^6 + 15750u_1^8 + 73991u_2^6u_1^2 \\ &- 2851u_1^2u_2^2 + 23658u_2^4u_1^2 - 29957u_2^4u_1^4 + 9791u_1^4u_2^2 - 4163u_2^2u_1^6 \\ &+ 69174u_2^6u_1^6 - 125788u_2^8u_1^2 - 48997u_2^{10}u_1^2 + 274u_2^8u_1^4 + 89942u_2^6u_1^4 \\ &- 22516u_2^4u_1^6 - 12163u_1^8u_2^2 + 36971u_1^8u_2^4 + 13567u_2^2u_1^{10} + 1031u_2^{12}u_1^4 \\ &- 1974u2^{12}u_1^2 - 2245u_2^{10}u_1^4 + 1717u_2^{10}u_1^6 - 5609u_2^6u_1^8 - 1052u_2^8u_1^6 \\ &+ 995u_2^8u_1^8 - 7766u_2^4u_1^{10} - 875u_2^4u_1^{12} - 3427u_1^{12}u_2^2 - 445u_2^6u_1^{10} \\ &- 409u_1^{14}u_2^2 + 407u_2^{12} - 1643u_2^{10} - 11592u_1^{10} - 15u_2^{14} - 504u_1^{14}; \end{aligned}$$

PROVIDED THAT :

$$u_1 \neq 0,$$

$$u_2 \neq 0,$$

$$(u_1+1)^2 + u_2^2 \neq 0,$$

$$(u_1-1)^2 + u_2^2 \neq 0,$$

$$\begin{aligned} L(u_1,u_2) = &\,9 + 84u_2^6u_1^2 + 84u_2^2 - 36u_1^2 - 116u_1^2u_2^2 + 54u_1^4 + 166u_2^4 - 140u_2^6 \\ &+ 132u_2^4u_1^2 + 25u_2^8 + 102u_2^4u_1^4 - 36u_1^6 + 9u_1^8 - 20u_1^4u_2^2 + 52u_2^2u_1^6 \neq 0, \end{aligned}$$

$$R1 \neq 0.$$

The total time for executing the five instructions is 87.69 seconds.

The non-degenerate condition $u_2 \neq 0$ is a premise because otherwise the vertices A, B, C are on a line. Thus $(u_1+1)^2+u_2^2 \neq 0$ and $(u_1-1)^2+u_2^2 \neq 0$

are verified. Furthermore, it can be easily shown (by DISCOVERER, say) that $L(u_1, u_2)$ is positive if $u_1 \neq 0$ and $u_2 \neq 0$. Because we are concerning the complement of the algebraic curve $R1 = 0$, the only "non-degenerate" condition we need to consider is $u_1 \neq 0$.

As we did in the preceding example, by typing in

Tofind($[R2, f, i], [\,], [-R1, x, x^2 - y^2], [u_1, u_2], [x, y], [u_1, u_2], 1$);
Tofind($[R2, f, i], [\,], [-R1, x, x^2 - y^2], [u_1, u_2], [x, y], [u_1, u_2], 3$);

we get the situation when (u_1, u_2) is on $R2 = 0$. Finally, we obtain

(1) If $u_1 \neq 0$,

The system has 3 (distinct) real solutions IF AND ONLY IF

$$[R1 < 0, R2 > 0, R3 < 0]$$

The system has 2 (distinct) real solutions IF AND ONLY IF

$$[R1 > 0]$$

The system has 1 (distinct) real solution IF AND ONLY IF

$$[R1 < 0, R2 \leq 0]$$

or

$$[R1 < 0, R2 > 0, R3 > 0]$$

The system does not have 0 or 4 real solution(s);

(2) If $u_1 = 0$ (ABC is an isosceles triangle),

The system has 2 (distinct) real solutions IF AND ONLY IF

$$[S1 \cdot S2 \geq 0]$$

The system has 1 (distinct) real solution IF AND ONLY IF

$$[S1 < 0, S2 > 0]$$

The system does not have 0 or 3 or 4 real solution(s)

where $S1 = u_2^4 - 22u_2^2 - 7,\ S2 = u_2^2 - 1/3$.

Note that if $u_1 = 0$ and the system has two distinct real solutions, then one of the solutions is of multiplicity 2 and thus the system has three real solutions indeed.

This example was studied in a different way by Guergueb *et al.*[18]. They did not give quantifier-free formulas but illustrated the situation with a sketch figure.

Example 55: Give the necessary and sufficient condition for the existence of a triangle with elements a, h_a, R, where a, h_a, R means the side-length, altitude, and circumradius, respectively.

Clearly, we need to find the necessary and sufficient condition for the following system to have real solution(s),

$$\begin{cases} f_1 = a^2h_a^2 - 4s(s-a)(s-b)(s-c) = 0, \\ f_2 = 2Rh_a - bc = 0, \\ f_3 = 2s - a - b - c = 0, \\ a > 0, b > 0, c > 0, a + b - c > 0, b + c - a > 0, \\ c + a - b > 0, R > 0, h_a > 0. \end{cases}$$

By the same way as in the preceding examples, we obtain

The system has real solution(s) IF AND ONLY IF

$$[0 \le R1, 0 \le R3]$$

or

$$[0 \le R1, R2 \le 0, R3 \le 0]$$

where

$$R1 = R - \frac{1}{2}a$$

$$R2 = Rh_a - \frac{1}{4}a^2$$

$$R3 = -\frac{1}{2}h_a^2 + Rh_a - \frac{1}{8}a^2.$$

The time spent is 0.61 seconds.

The condition given by Mitrinovic *et al.*[27] is $R1 \ge 0 \wedge R3 \ge 0$. Now, we know they are wrong and that is only a sufficient condition.

Our program, DISCOVERER, is very efficient for solving this kind of problems. By DISCOVERER, we have discovered or rediscovered about 70 such conditions for the existence of a triangle, and found three mistakes in Mitrinovic *et al.* [27].

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