Real Solution Classification for Parametric Semi-Algebraic Systems

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Abstract

Real solution classification of parametric polynomial systems is a crucial problem in real quantifier elimination which interests Volker Weispfenning and others. In this paper, we present a stepwise refinement algorithm for real solution classifications of a class of parametric systems consisting of polynomial equations, inequalities and inequations. For an input system, the algorithm outputs the necessary and sufficient conditions (in terms of quantifier-free formulae) on the parameters for the system to have a given number of real solutions. Although the algorithm makes use of a pcad algorithm, it is different from any existing methods.

1 Introduction

Let us begin with a simple example [MPV89]: Give the necessary and sufficient conditions for the existence of a triangle with elements a, h_a and R, where a, h_a and R are the side-length, altitude, and circumradius, respectively. Let b and c be the other two side-lengths and s the half perimeter, the problem is reduced to finding the necessary and sufficient conditions for the following system to have real solutions (s, b and c are viewed as variables and a, h_a and R parameters).

$$(\mathrm{AHR}):\begin{cases} p_1 = a^2h_a^2 - 4s(s-a)(s-b)(s-c) = 0, \\ p_2 = 2Rh_a - bc = 0, \\ p_3 = 2s - a - b - c = 0, \\ a > 0, b > 0, c > 0, a + b - c > 0, b + c - a > 0, \\ c + a - b > 0, R > 0, h_a > 0. \end{cases}$$

This is a problem of real quantifier elimination and the method of cylindrical algebraic decomposition (cad) [Col75, McC88, CH91, Hong92, CJ98, McC99, Br00, Br01] is a well-known tool for this kind of problems. Also, it is well-known that real quantifier elimination is one of the main interests of Volker Weispfenning. He has made many contributes to this field. See, for example, [Weis94, Weis97, DSW98a, DSW98b, DW00, SW02].

We want to obtain a quantifier-free solution formula which only consists of the signs of polynomials in a, R and h_a. So, we solve this problem in a different way as follows. First, we compute the so-called *border polynomial* (see next section for the definition), *BP*, of the system. For this example,

$$BP_{\text{AHR}} = aRh_a(2R-a)(2R+a)(8Rh_a-4h_a^2-a^2)(8Rh_a+4h_a^2+a^2).$$

An essential property of this polynomial is that the number of distinct real solutions of the system (AHR) is invariant in each connected component of the complement of $BP_{\text{AHR}} = 0$ in $\mathbb{R}^3$. If we discard the factors of BP_{AHR} which have no real roots under the constraints, we need only to consider two factors: $q_1 = 2R-a$ and $q_2 = 8Rh_a-4h_a^2-a^2$.

Generally speaking, the signs of the factors of *BP* only determine some necessary but not sufficient conditions for the system under discussion to have real solutions. For this example, on one hand, $q_1 \geq 0$ is a necessary condition for the system (AHR) to have real solutions. On the other hand, if $a = 1, R = 1, h_a = 2$, then $q_1 > 0$ and $q_2 < 0$ and the system (AHR) has no real roots. And, if $a = 1, R = 1, h_a = 1/10$, then $q_1 > 0$ and $q_2 < 0$ also hold but the system has four distinct real solutions. That is to say the necessary and sufficient conditions we want can not be determined by the signs of the factors of BP_{AHR}.

Second, we multiply *BP* by some other polynomials to construct a new polynomial *PS* (see next section for details). For this example, we obtain

$$PS_{\text{AHR}} = BP_{\text{AHR}} \cdot (a+2h_a)(a-2h_a)(4Rh_a-a^2)(4Rh_a+a^2).$$

And the signs of the factors of PS_{AHR} determine the answer to the problem. In fact, letting $q_3 = 4Rh_a - a^2$, we obtained that the system (AHR) has real solutions if and only if

$$(q_1 \geq 0 \wedge q_2 \geq 0) \ \vee \ (q_1 \geq 0 \wedge q_2 \leq 0 \wedge q_3 \leq 0).$$

Actually, because the number of distinct real solutions of the system is invariant in each connected component of the complement of $BP = 0$, we can obtain the so-called *real solution classification* of the system, i.e., the necessary and sufficient conditions for the system to have a given number of real solutions.

The rest of this paper is organized as follows. Section 2 introduces the definitions of *border polynomial* and *discrimination polynomial* of a given semi-algebraic system (SAS). The idea of Section 2 suggests naturally an algorithm for real solution classification of a SAS, which will be described in Section 3. Section 4 comments on our present work.

2 Main Idea

we call

$$[[P], [G_1], [G_2], [H]] \tag{1}$$

a *semi-algebraic system* (SAS for short), where P, G_1, G_2 and H denote $\{p_1(x_1,...,x_n) = 0, ..., p_s(x_1,...,x_n) = 0\}$, $\{g_1(x_1,...,x_n) \geq 0, ..., g_r(x_1,...,x_n) \geq 0\}$, $\{g_{r+1}(x_1,...,x_n) > 0, ...,$

$g_t(x_1,...,x_n) > 0\}$ and $\{h_1(x_1,...,x_n) \neq 0,...,h_m(x_1,...,x_n) \neq 0\}$, respectively. Here, $n, s \geq 1$, $r,t,m \geq 0$ and p_i, g_j, h_k are all polynomials in $x_1,...,x_n$ with integer coefficients. A SAS is called a *parametric* SAS if $s < n$ (s indeterminates are viewed as independent variables and the other $n-s$ indeterminates parameters).

There exist several famous methods, such as the Ritt-Wu method, Gröbner basis method or subresultant method [Wu78, Buch85, YZH92, Wang98], which enable us to transform a given equations into equations with special structures. Throughout this paper, we assume that the parametric SAS in the form of (1) can be transformed into one or more systems in the form of

$$[[F],[G_1],[G_2],[H]] \tag{2}$$

where $F = \{f_1(u,x_1), f_2(u,x_1,x_2),..., f_s(u,x_1,x_2,...,x_s)\}$ is a *normal ascending chain* [YZH92] (or a *regular chain* by [Ka93] and a *regular set* by [Wang00]). We call a system in the form of (2) a *parametric* TSA. In a future paper, we will discuss how to deal with general SASs in the form of (1) by a method similar to that given in this paper.

For a given parametric SAS defined by (1), let $(x_{s+1},...,x_n)$ be denoted by $u = (u_1,...,u_d)$ and be viewed as parameters. Let $Q^* = \{q_i(u) \in \mathbb{Z}[u_1,\cdots,u_d] | 1 \leq i \leq l\}$ be a finite set of polynomials in parameters and Q denote the product of all the elements of Q^*. For each i $(1 \leq i \leq l)$ and each connected component C of the complement of $Q = 0$ in $\mathbb{R}^d$, $\text{sign}(q_i)$ is invariant in C and is not zero. For $\overline{u} \in C$, we call $[\text{sign}(q_1(\overline{u})),...,\text{sign}(q_l(\overline{u}))]$ the *sign* of C. Obviously, each connected component of the complement of $Q = 0$ has a unique sign but two different components may have the same sign (So, the *sign* of a component is generally different from the *defining formula* of the component).

Theorem 1 *Let a parametric* SAS *S be given. Suppose a polynomial Q in u satisfies that*

- *(a) the number of distinct real solutions of S is invariant in each connected component of the complement of $Q = 0$ in $\mathbb{R}^d$ and*
- *(b) if two components, C_1 and C_2, have the same sign, the number of distinct real solutions of S in C_1 equals that of S in C_2.*

If we only consider the complement of $Q = 0$ in the parametric space, the necessary and sufficient conditions for S to have exactly N distinct real solution(s) can be expressed by the signs of the factors of Q. If Q satisfies item (a) only, then some necessary conditions for S to have exactly N distinct real solution(s) can be expressed by the signs of the factors of Q.

Although Theorem 1 seems obvious, it provides an idea for determining the real solution classification of a SAS. In the rest of this section, we discuss how to construct a polynomial in parameters satisfying the two conditions of Theorem 1 for a given parametric SAS.

Given a polynomial q and a triangular set $\{f_1, f_2,..., f_s\}$, we define

$$\text{res}(q; f_i,..., f_1) = \text{res}(\cdots(\text{res}(\text{res}(q, f_i,x_i), f_{i-1},x_{i-1}),...), f_1,x_1),$$

where $\text{res}(p,q,x)$ is the *Sylvester resultant* of p and q with respect to x. Let the leading coefficient and the discriminant of a polynomial f with respect to x be denoted by $\text{lc}(f,x)$ and $\text{dis}(f,x)$, respectively.

Definition 2 For a parametric TSA T, we define

$$BP_T = BP = \begin{array}{l} \mathrm{lc}(f_1,x_1)\cdot \mathrm{dis}(f_1,x_1)\cdot \\ \prod_{2\le i\le s}\mathrm{res}(\mathrm{lc}(f_i,x_i)\cdot \mathrm{dis}(f_i,x_i); f_{i-1},\dots,f_1)\cdot \\ \prod_{1\le j\le t}\mathrm{res}(g_j; f_s,\dots,f_1)\cdot \\ \prod_{1\le k\le m}\mathrm{res}(h_k; f_s,\dots,f_1), \end{array} \quad (3)$$

and call it the *border polynomial* of T. A TSA is *regular* if $BP \not\equiv 0$.

Remark 3 For a regular TSA T, if we only consider the complement of $BP = 0$ in $\mathbb{R}^d$, T can be regarded as a system in the form of $[[F],[\,],[G],[\,]]$, i.e.,

$$\begin{cases} f_1(u,x_1) = 0,\dots,f_s(u,x_1,\dots,x_s) = 0, \\ g_1(u,x_1,\dots,x_s) > 0,\dots,g_t(u,x_1,\dots,x_s) > 0 \end{cases} \quad (4)$$

because $\mathrm{res}(g_j; f_s,\dots,f_1) \neq 0$ and $\mathrm{res}(h_k; f_s,\dots,f_1) \neq 0$ imply that $g_j \neq 0$ and $h_k \neq 0$ at the solutions of $\{f_1 = 0,\dots,f_s = 0\}$.

Remark 4 We gave an algorithm [YHX01] for decomposing any TSA (or SAS) into regular TSAs based on the so-called RSD algorithm [YZH92]. So, we mainly discuss on regular TSAs in the form of (4).

Suppose $T_1 : [[F^{(1)}],[\,],[G],[\,]]$ and $T_2 : [[F^{(2)}],[\,],[G],[\,]]$ are two regular parametric TSAs in the form of (4). If $r_1 = \mathrm{res}(f_1^{(1)}, f_1^{(2)}, x_1)$ is a nonzero integer, T_1 and T_2 have no common solutions. Otherwise, $r_1 = 0$ is a necessary and sufficient condition for $f_1^{(1)}$ and $f_1^{(2)}$ to have common roots in $\mathbb{C}$. Let

$$r_i = \gcd(\mathrm{res}(f_i^{(1)}; f_i^{(2)},\dots,f_1^{(2)}), \mathrm{res}(f_i^{(2)}; f_i^{(1)},\dots,f_1^{(1)})) \quad (2\le i\le s)$$

and $CP_{12} = \gcd(r_1,\dots,r_s)$. Without loss of generality, we can assume that $CP_{12} \not\equiv 0$ because that case can be removed by the RSD algorithm [YZH92]. If $CP_{12} \neq 0$, T_1 and T_2 have no common solutions.

Suppose a parametric SAS S is transformed equivalently to regular TSAs $T_1,\dots,T_l$, for every pair of (T_i,T_j) $(i \neq j)$, we can compute CP_{ij} analogously and then define $CP_S = \prod_{1\le i<j\le l} CP_{ij}$.

Definition 5 If a parametric SAS S is transformed equivalently to regular TSAs $T_1,\dots,T_l$, then $BP_S = CP_S \cdot \prod_{i=1}^{l} BP_{T_i}$ is called the *border polynomial* of S.

Theorem 6 *If T is a regular* TSA*, then BP satisfies item (a) in Theorem 1 for the system T. BP_S satisfies item (a) in Theorem 1 for a parametric* SAS *S.*

Suppose $g(x)$ and $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ are two real polynomials and $r(x) = \mathrm{rem}(f'g, f, x) = b_0x^{n-1} + b_1x^{n-2} + \cdots + b_{n-1}$. The following $2n \times 2n$ matrix is called the *generalized discrimination matrix* of $f(x)$ with respect to $g(x)$ and denoted by $\mathrm{Discr}(f,g)$.

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_n & & & \\ 0 & b_0 & b_1 & \cdots & b_{n-1} & & & \\ & a_0 & a_1 & \cdots & a_{n-1} & a_n & & \\ & 0 & b_0 & \cdots & b_{n-2} & b_{n-1} & & \\ & & & \cdots & \cdots & & & \\ & & & \cdots & \cdots & & & \\ & & & a_0 & a_1 & a_2 & \cdots & a_n \\ & & & 0 & b_0 & b_1 & \cdots & b_{n-1} \end{bmatrix}$$

Let $D_0 = 1$ and denote by $D_1, D_2, \cdots, D_n$ the even order principal minors of $\mathrm{Discr}(f,g)$. We call $[D_0, D_1, \cdots, D_n]$ the *generalized discriminant sequence* of $f(x)$ with respect to $g(x)$ and denote it by $\mathrm{GDL}(f,g)$.

Theorem 7 [GLRR89, YHZ96, Yang99] *Let two real polynomials $f(x)$ and $g(x)$ be given.*

(a) The number of distinct real zeros of f is determined by the signs of polynomials in $\mathrm{GDL}(f,1)$*;*

(b) The number of distinct real solutions of $\{f = 0, g > 0\}$ is determined by the signs of polynomials in $\mathrm{GDL}(f,1)$ *and* $\mathrm{GDL}(f,g)$.

The above theorem can be obtained from the theory of subresultants. The main idea of the proof is to establish the relations between the elements of $\mathrm{GDL}(f,g)$ and the leading coefficients of polynomials in standard Sturm sequence of f and fg'. According to the above theorem, the real root classification of $\{f = 0, g > 0\}$ is determined *explicitly* by the signs of polynomials in $\mathrm{GDL}(f,1)$ and $\mathrm{GDL}(f,g)$.

Let $A = \{A_i | 1 \le i \le l\}$ be a nonempty, finite set of polynomials. We define

$$\mathrm{mset}(A) = \{1\} \cup \{A_{i_1}A_{i_2}\cdots A_{i_k} | 1 \le k \le l, 1 \le i_1 < i_2 < \cdots < i_k \le l\}.$$

Given a regular TSA T in the form of (4), we define

$$\mathbf{P}_{s+1} = \{g_1, g_2, \cdots, g_t\}; \;\; \mathbf{P}_i = \bigcup_{q \in \mathrm{mset}(\mathbf{P}_{i+1})} \mathrm{GDL}(f_i, q), \quad \text{for } i = s, \cdots, 1,$$

where $\mathbf{P}_i$ is the set consisting of all the polynomials in each $\mathrm{GDL}(f_i, q)$ for $q \in \mathrm{mset}(\mathbf{P}_{i+1})$.

Definition 8 We denote the product of all elements in $\mathbf{P}_1$ by DP_T or DP if the meaning is clear and call it the *discrimination polynomial* of T. It is clear that BP_T divides DP_T. If a parametric SAS S is transformed equivalently to regular TSAs $T_1, ..., T_l$, then $DP_S = CP_S \cdot \prod_{i=1}^{l} DP_{T_i}$ is called the *discrimination polynomial* of S.

Theorem 9 *If T is a regular* TSA *in the form of* (4)*, then DP_T satisfies the two conditions of Theorem 1 for the system T. DP_S satisfies the two conditions of Theorem 1 for a parametric* SAS *S.*

3 The Algorithm

By Theorems 6 and 9, it is natural to propose a stepwise refinement algorithm as follows for real solution classification of a given parametric SAS S.

Step 1. We discuss the complement of $BP_S = 0$ of the parametric space. Let $PS = BP_S$. It is natural to employ a PCAD algorithm to obtain sample points in each connected component of the complement of $PS = 0$ and compute the number of distinct real solutions of S at each sample point. Then, we compute the *sign* of each component. Note that the *defining formula* of a cell by PCAD may be very complex while the *sign* of the cell is usually simple. If the second property of Theorem 1 are not satisfied by the present PS, choose some polynomials (factors) from DP_S, multiply PS by these polynomials and repeat the above procedure. Obviously, the procedure will terminate within a finite steps (at most when $PS = DP_S$) and output the real solution classification of S when the parameter is in the complement of $BP_S = 0$.

Step 2. Let BP_S^* be the set of irreducible factors of BP_S and $R(u_1, \ldots, u_d) \in BP_S^*$. We denote by SR the new parametric SAS formed by adding $R = 0$ into S. Regarding $(u_1, x_1, \ldots, x_s)$ as variables and $(u_2, \ldots, u_d)$ parameters, we can compute the border polynomial $BP_{SR}(u_2, \ldots, u_d)$. Thus, we can take use of a procedure similar to Step 1 to obtain the real solution classification of S when the parameters satisfy $R = 0$ and $BP_{SR} \neq 0$. We can call this procedure again, inputting the system SR and a new "boundary" $Q \in BP_{SR}^*$, to obtain the real solution classification of S when the parameters are on $R = 0$ and $Q = 0$ provided $BP_{SRQ} \neq 0$. It is easy to see that we can repeat this procedure, adding a new "boundary" each time, until the complete classification of real solutions of S on $R = 0$ is obtained. Note that, at the final stage, if the equations in the parameters $(u_1, \ldots, u_d)$ give only a finite points in $\mathbb{R}^d$, we need to call an isolation algorithm [XY02] to isolate the real solutions of the system.

Now, combining Steps 1 and 2, we have a stepwise refinement algorithm for the complete classification of real solutions of a parametric SAS in $\mathbb{R}^d$. For the complement of $BP = 0$ in the parametric space, Step 1 is enough and the solution formula only consists of the signs of some parametric polynomials. If $BP = 0$ need to be considered, the real roots of $BP = 0$ is needed of course.

The algorithm has been implemented as a Maple program "DISCOVERER" which computed many examples including the one in Section 1 of the present paper. Interested readers may request the code from the second author.

4 Conclusions

For a semi-algebraic system (SAS) satisfying some conditions, we define the border polynomial and discrimination polynomial and present an effective algorithm for real solution classification of the system. It is not difficult to generalize the algorithm to deal with other types of SASs such as the Whiteney Umbrella problem and the Solotareff problem. That will be clarified in a future paper.

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