Filled Function Methods for Global Optimization Problems

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Outline

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1. Preliminaries

Consider the following global optimization problem:

$$(GP) \quad \min \quad f(x)$$
$$x \in X,$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and $X \subset \mathbb{R}^n$.

- If $X = \mathbb{R}^n$, then problem (GP) is an unconstrained global optimization problem.
- If $X \subset \mathbb{R}^n$, especially

  $$X := \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, i = 1, \ldots, m\},$$

  then problem (GP) is a constrained global optimization problem.

- For problem (GP), there exist many mature local optimization methods to obtain a local minimizer x^* in literature.
- Our aim is to find a global minimizer.
- Among all the different types of global optimization algorithms available in the literature, one popular approach is called modified function method.
- Filled function method is a typical modified function method.
- The main idea of filled function method is as followings.

  Let x^* be a local minimizer and not a global minimizer of problem (GP). Then construct a filled function $p_{x^*}(x)$ at x^* such that

1°. the local minimizer x^* of problem (GP) is a strictly local maximizer of function $p_{x^*}(x)$ on X;

2°. we can escape the current point x^* to obtain a **better point** by solving the filled function problem $\min_{x\in X} p_{x^*}(x)$ via using some local methods.

See the Figure.

- How to construct a good filled function to satisfy the above conditions?
- This talk will introduce some new filled functions and some new filled function methods for global optimization problems and nonlinear equations.
- References:

[WLZY]: Z.Y. Wu, H.W.J. Lee, L.S. Zhang and X. M. Yang, A novel filled function method and quasifilled function method for global optimization, *Journal of Computational Optimization and Applications*, 34(2), 249-272, 2005.

[WHBY]: Z.Y. Wu, H.W.J. Lee, F.S. Bai and Y.J. Yang, A filled function methods for constrained global optimization, *Journal of Global Optimization*,39(4), 495-507, 2007.

[WMBY]: Z.Y. Wu, M. Mammadov, F.S. Bai and Y.J. Yang, A Filled Function Method for Nonlinear Equations, *Applied Mathematics and Computation*, 189, 1196-1204, 2007.

2. Filled function method for unconstrained global optimization problems

Consider the following unconstrained programming problem:

$$(UGP) \quad \min \quad f(x)$$
$$\text{s.t.} \quad x \in R^n,$$

where $f(x)$ is continuously differentiable on R^n.

- Reference $[WLZY]$ propose a new filled function for problem (UGP). To introduce this new filled function, we need two auxiliary functions.

  $\triangledown$ Let

$$f_r(t) = \begin{cases} t+r & t \leq -r \\ \dfrac{r-2}{r^3}t^3 + \dfrac{r-3}{r^2}t^2 + 1, & -r < t \leq 0 \\ 1 & t > 0 \end{cases} . \tag{2.1}$$

$$g_r(t) = \begin{cases} 0, & t \leq -r \\ -\dfrac{2}{r^3}t^3 - \dfrac{3}{r^2}t^2 + 1, & -r < t \leq 0 \\ 1, & t > 0 \end{cases} . \tag{2.2}$$

– Functions $f_r(t)$ and $g_r(t)$ are continuously differentiable on R, see Figure 2.1 and Figure 2.2.

▽ Let

$$H_{q,r,x^*}(x) = q\Bigg(\exp(-\frac{\|x-x^*\|^2}{q}) g_r\Big(f(x)-f(x^*)\Big) + f_r\Big(f(x)-f(x^*)\Big)\Bigg), \qquad (2.3)$$

where $r > 0,\ q > 0$ are parameters, x^* is the current local minimum and $exp(.)$ is an exponential function.

- Function H_{q,r,x^*} has the following properties:
  - ♦ Suppose x^* is a local minimizer of problem (UGP), then x^* is a strictly local maximizer of $H_{q,r,x^*}(x)$ on $\mathbb{R}^n$ for any $r > 0,\ q > 0$.
  - ♦ Any local minimizer $\bar{x}$ of $H_{q,r,x^*}(x)$ on $\mathbb{R}^n$ satisfies that $f(\bar{x}) < f(x^*)$.

♦ Suppose that x^* is not a global minimizer of problem (UGP). Let

$$L = \{\bar{x} \mid \bar{x} \text{ is local minimizer satisfying } f(\bar{x}) < f(x^*)\}.$$

Then for any $\bar{x} \in L$, $\bar{x}$ is a local minimizer of $H_{q,r,x^*}(x)$ on $\mathbb{R}^n$ when the parameters r satisfies some conditions.

■ From above properties, we know that: x^* **is a strictly local maximizer of** $H_{q,r,x^*}(x)$ **on** $\mathbb{R}^n$ and **if** x^* **is not a global minimizer of problem** (UGP)**, then any local minimizer** $\bar{x}$ **of** $H_{q,r,x^*}(x)$ **on** $\mathbb{R}^n$ **is a better point, i.e.,** $f(\bar{x}) < f(x^*)$. Thus, function $H_{q,r,x^*}(x)$ is a filled function of problem (UGP) at x^*.

- Using the given filled function $H_{q,r,x^*}(x)$, we can design a global optimization method for problem (UGP).

Algorithm 1: Filled function method for Problem (UGP):

Step 0. Let k_0 be a positive number and let $e_i,\ i = 1, \cdots, k_0$ almost uniformly distribute over the unit sphere $B = \{x \in R^n \mid \|x\| = 1\}$. Let M be a very large number and μ be a very small number. Choose an initial point $x_1^0 \in \mathbb{R}^n$. Set $k := 1$.

Step 1. Let x_k^* be a local minimizer of problem (UGP) starting from x_k^0. Set $i := 1$ and take a positive number $\delta_0 > 0$, let $\delta := \delta_0$.

Step 2. Let $\bar{x}_k^* = x_k^* + \delta e_i$. If $f(\bar{x}_k^*) < f(x_k^*)$, then set $x_{k+1}^0 := \bar{x}_k^*,\ k := k + 1$ and go to *Step 1*; otherwise, go to *Step 3*.

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Step 3. Let

$$H_{q,r,x_k^*}(x) = q\Bigg(\exp(-\frac{\|x - x_k^*\|^2}{q}) g_r\Big(f(x) - f(x_k^*)\Big) + f_r\Big(f(x) - f(x_k^*)\Big)\Bigg),$$

where $g_r(t)$ and $f_r(t)$ are decided by (2.2) and (2.1), respectively. Solve the problem:

$$\begin{array}{ll} \min & H_{q,r,x_k^*}(x) \\ & x \in \mathbb{R}^n \end{array} \tag{2.4}$$

by a local search method starting from the point $\bar{x}_k^*$. If we can find a local minimizer y_k^*, then we have that $f(y_k^*) < f(x_k^*)$, then let $x_{k+1}^0 := y_k^*$ and goto *step 1*; otherwise, goto *Step 4*. We have two cases: one is that problem (2.4) has no local minimizer, then x_k^* is already a global minimizer; another one is that (2.4) has local minimizer, but we can not find them, then we need to change the direction e_i,or change the parameters q or r.

Step 4. If $i < k_0$, then let $i := i + 1$, go to *Step 2*; otherwise, go to *Step 5*.

Step 5. If $q < M$ and $r > \mu$, increase q and decrease r; otherwise, go to *Step 6*.

Step 6. Stop and x_k^* is a global minimizer of problem (UGP).

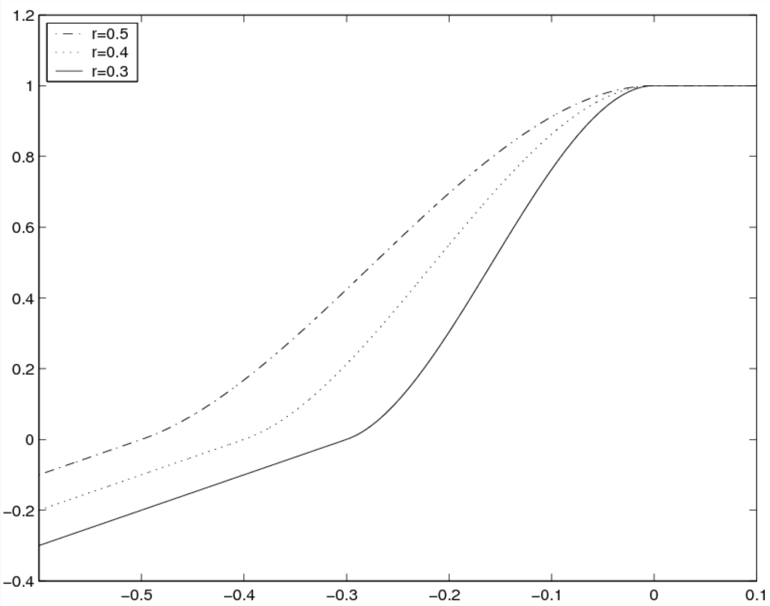

Figure 2.1: The behavior of $f_r(t)$ with $r = 0.5,\ 0.4$ and 0.3, respectively

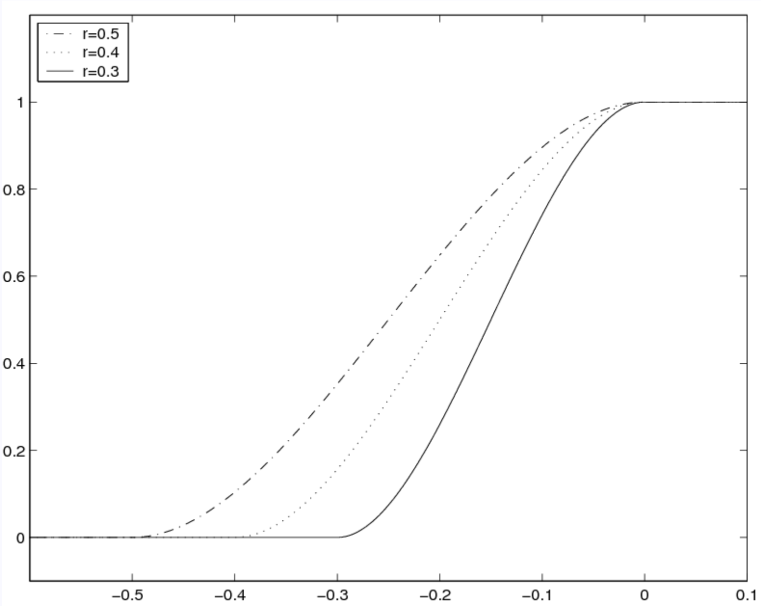

Figure 2.2: The behavior of $g_r(t)$ with $r = 0.5,\ 0.4$ and 0.3, respectively

3. Filled function method for constrained global optimization problems

Consider the following constrained global optimization problems:

$$
\begin{aligned}
(CGP) \qquad \min\ & f(x) \\
\text{s.t. } & g_i(x) \le 0,\ i = 1, \ldots, m \\
& x \in X,
\end{aligned}
\tag{3.1}
$$

where $f : X \to R$, $g_i : X \to R, i = 1, 2, \ldots, m$, are continuously differentiable on X, X is a box.

- Let

$$
\begin{aligned}
S &= \{x \in X \mid g_i(x) \le 0,\ i = 1, \ldots, m\}, \\
S^\circ &= \{x \in \mathrm{int}X \mid g_i(x) < 0,\ i = 1, \ldots, m\},
\end{aligned}
$$

where $\mathrm{int}A$ denotes the interior of set A.

Assumption 1. Assume that $S^\circ \neq \emptyset$, $\mathrm{cl}S^\circ = S$, where $\mathrm{cl}A$ denotes the closure of set A.

□ By Assumption 1, we know that for any $x_0 \in S$, there exists a sequence $\{x_n\} \subset S^\circ$, such that $\lim_{n\to\infty} x_n = x_0$.

- Reference $[WHBY]$ proposes a filled function method to solve problem (CGP). Here we also need the following two auxiliary functions.

△ For $r > 0, c > 0$, let

$$f_{r,c}(t) = \begin{cases} c, & t \geq 0 \\ -\frac{2c}{r^3}t^3 - \frac{3c}{r^2}t^2 + c, & -r < t \leq 0 \\ 0, & t \leq -r \end{cases}, \tag{3.2}$$

$$h_r(t) = \begin{cases} t+2 & t \geq 0 \\ \dfrac{r-4}{r^3}t^3 + \dfrac{2r-6}{r^2}t^2 + t + 2 & -r < t < 0 \\ 0 & t \leq -r \end{cases} . \tag{3.3}$$

△ Let

$$p_{r,c,q,x^*}(x) = \frac{1}{\|x-x^*\|^2+1} f_{r,c}\Bigg(h_r\Big(f(x) - f(x^*)\Big) + \sum_{i=1}^{m} h_{\frac{r}{q}}\Big(g_i(x)\Big) - 2r \Bigg), \tag{3.4}$$

where $c > 0$, $r > 0$ and $q > 0$ are parameters.

□ The term $\sum_{i=1}^{m} h_{\frac{r}{q}}\Big(g_i(x)\Big)$ is used to penalize the unfeasible points. The term $h_r\Big(f(x) - f(x^*)\Big)$ is used to penalize the points x which satisfy that

$f(x) \geq f(x^*)$. So here the function $p_{r,c,q,x^*}(x)$ is not only a filled function, but also a penalty function for problem (CGP).

△ Function $p_{r,c,q,x^*}(x)$ has the following properties.

* If x^* is a local minimizer of problem (CGP), then for any $c > 0$, $q > 0$ and $0 < r \leq 1$, x^* is a strictly local maximizer of $p_{r,c,q,x^*}(x)$ on X.
* For any $x \in X$ with $x \neq x^*$, if $\nabla p_{r,c,q,x^*}(x) = 0$, then $f(x) < f(x^*)$ and $x \in S$.
* When $r \leq 1$, any local minimizer $\bar{x}$ of $p_{r,c,q,x^*}(x)$ on X satisfies that

$$f(\bar{x}) < f(x^*) \text{ and } \bar{x} \in S,$$

or

$$\bar{x} \text{ is a vertex of } X.$$

* If x^* is not a global minimizer of problem (CGP),

then there exist $r_0 > 0$, $q_0 > 0$ and $\bar{x} \in S^\circ$, such that $\bar{x}$ is a local minimizer of $p_{r,c,q,x^*}(x)$ on X and $f(\bar{x}) < f(x^*)$ when $r \leq r_0$ and $q \geq q_0$.

- By the given filled function $p_{r,c,q,x^*}(x)$, the local optimization methods for constrained problem (CGP) and the local optimization methods for the unconstrained filled function problems(with only box constraint): $\min_{x \in X} p_{r,c,q,x^*}(x)$, we can obtain the following global optimization method for problem (CGP).

Algorithm 2: Filled function method for Problem (CGP):

Step 0. a). Choose a small positive numbers μ, and a large positive number M. Choose a positive integer number K and directions $e_1, \ldots, e_K$. Choose the initial values q_1, c_1, and r_1 for the parameters q, c, and r, respectively.

b). Choose an initial point $x_1^0 \in X$(here x_1^0 may not be a feasible point), then use penalty function methods to find the first local minimizer x_1^* of the original problem (CGP). Let $k := 1, j := 1$ and $\lambda := 1$, and go to *Step 1*.

Step 1. Let

$$p_{r_k,c_k,q_k,x_k^*}(x) = \frac{1}{\|x - x_k^*\|^2 + 1} f_{r_k,c_k}\Bigg(g_{r_k}\Big(f(x) - f(x_k^*)\Big) + \sum_{i=1}^{m} g_{\frac{r_k}{q_k}}\Big(g_i(x)\Big) - 2r_k \Bigg), \tag{3.5}$$

where $f_{r,c}(t)$ and $g_r(t)$ are defined in (3.2) and (3.3) respectively. Go to *Step 2*.

Step 2. If $j \leq K$, choose a nonnegative λ with $\lambda \leq 1$ such that $y_k^j := x_k^* + \lambda e_j \in X$, and go to *Step 3*; otherwise, go to *Step 5*.

Step 3. Search for a local minimizer of the following filled function problem starting from y_k^j:

$$\min_{x \in X} \quad p_{r_k, c_k, q_k, x_k^*}(x). \tag{3.6}$$

Let $\bar{x}_k^*$ be an obtained local minimizer of problem (4.3). If $\bar{x}_k^*$ satisfies $f(\bar{x}_k^*) < f(x_k^*)$ and $\bar{x}_k^* \in S$, then let $x_{k+1}^0 := \bar{x}_k^*, k := k + 1$, and go to $Step\ 4$; otherwise, let $j := j + 1$ and go to *Step 2*.

Step 4. Find a local minimizer x_k^* of the original constrained problem (CGP) by local search methods starting from x_k^0. Go to $step\ 1$.

Step 5. If $q_k \leq M$, increase q_k and let $j := 1$, go to *Step 1*;

otherwise, go to $Step\ 6$.

Step 6. If $c_k \leq M$, increase c_k and let $q_k := q_1$, $j := 1$, go to *Step 1*; otherwise, go to *Step 7*.

Step 7. If $r_k \geq \mu$, decrease r_k and let let $c_k := c_1, q_k := q_1$, go to *Step 1*; otherwise, stop and x_k^* is a global minimizer or an approximate global minimizer of problem (CGP).

4. Filled function method for Nonlinear Equations

Consider the following nonlinear equations:

$$(NE) \qquad F(x) = 0$$
$$x \in X,$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and $X \subset \mathbb{R}^n$.

- The typical methods for solving (NE) are optimization-based methods in which (NE) is reformulated as an optimization problem.
- The most popular optimization-based methods involve solving the following optimization problem (OP) to

find solutions of equations (NE).

$$(OP) \qquad \min \ \varphi(x) := \frac{1}{2}\|F(x)\|_2^2$$
$$\text{s.t.} \quad x \in X.$$

♡ If (NE) exists a solution in X, then $\bar{x} \in X$ is a solution of (NE) if and only if $\bar{x}$ is a global optimal solution of problem (OP) with the zero optimal value.

♡ Generally, the traditional optimization-based methods for solving nonlinear system (NE) are frequently stuck at a stationary point or a local minimizer of the corresponding optimization problem, which is not necessarily a solution of the original system.

♡ Recently, great efforts have been made to overcome the difficulty caused by non-global minimizers.

♡ Reference: (C. Kanzow, "Global optimization techniques for mixed complementarity problems," *Journal of Global Optimization*, vol. 16, pp. 1-21, 2000) incorporated two well-known global optimization algorithms, namely a tunneling method and a filled function method, into a standard nonsmooth Newton-type method to solve an unconstrained nonsmooth nonlinear system (NE) which is a reformulation of the mixed complementarity problem.

♡ Reference [WMBYE] proposes a new filled function method to solve nonlinear equations (NE) with box constraint. Here I will introduce this method. First, we need several definitions.

▲ A point $x \in X$ is said to be a *vertex* of box X

if $x = \lambda x_1 + (1-\lambda)x_2$ with $x_1, x_2 \in X$ and $\lambda \in (0,1)$ implies that $x = x_1 = x_2$.

▲ **Definition of Filled Function for (NE):** A continuously differentiable function $P_{x^*}(x)$ is said to be a filled function of (NE) at a point x^* with $\varphi(x^*) > 0$, if:

1° x^* is a strict local maximizer of $P_{x^*}(x)$ on X;

2° Any local minimizer $\bar{x}$ of $P_{x^*}(x)$ on X satisfies

$$\varphi(\bar{x}) < \frac{\varphi(x^*)}{2} \quad \text{or} \quad \bar{x} \text{ is a vertex of } X;$$

3° Any local minimizer $\bar{x}$ of problem (OP) with $\varphi(\bar{x}) \le \frac{\varphi(x^*)}{4}$ is a local minimizer of $P_{x^*}(x)$ on X;

4° Any $\bar{x} \in X$ with $\nabla P_{x^*}(\bar{x}) = 0$ implies $\varphi(\bar{x}) < \frac{\varphi(x^*)}{2}$.

★ Note that in the definition of filled function for (NE), it is not necessary to require that x^* is a local minimizer of optimization problem (OP). It just needs that $\varphi(x^*) > 0$

▲ **Construct a Filled Function for (NE):** Using the two auxiliary functions $g_r(t)$ and $f_r(t)$ defined by (2.2) and (2.1), for a given $x^* \in X$ with $f(x^*) > 0$, let

$$\Psi_{q,x^*}(x) = \frac{1}{\|x - x^*\|^2 + 1}\left(g_{\frac{\varphi(x^*)}{4}}\left(\varphi(x) - \frac{\varphi(x^*)}{2}\right)\right) + q\, f_{\frac{\varphi(x^*)}{4}}\left(\varphi(x) - \frac{\varphi(x^*)}{2}\right). \tag{4.1}$$

• **Function $\Psi_{q,x^*}(x)$ has the following properties:**

△ Let $\varphi(x^*) > 0$, $q > 0$. Then x^* is a strict global maximizer of $\Psi_{q,x^*}(x)$ on X.

- △ Let $\varphi(x^*) > 0,\ q > 0$. Any local minimizer $\bar{x}$ of $\Psi_{q,x^*}(x)$ on X satisfies

$$\varphi(\bar{x}) < \frac{\varphi(x^*)}{2} \quad \text{or} \quad \bar{x} \text{ is a vertex of } X.$$

- △ Let $\varphi(x^*) > 0,\ q > 0$. Assume that system (NE) has a solution. Then any local minimizer $\bar{x} \in X$ of problem (OP) on X with $\varphi(\bar{x}) < \frac{\varphi(x^*)}{4}$ is a local minimizer of $\Psi_{q,x^*}(x)$ on X.
- △ Let $\varphi(x^*) > 0$. Then any point $\bar{x} \in X \setminus \{x^*\}$ with $\nabla\Psi_{q,x^*}(\bar{x}) = 0$ implies that $\varphi(\bar{x}) < \frac{\varphi(x^*)}{2}$.
- ★ Function $\Psi_{q,x^*}(x)$ is a filled function of (NE). Using this filled function, we can obtain the following method to solve (NE).

Algorithm 3: Filled Function Method for Nonlinear equations (NE):

Step 0. Choose small positive numbers μ, δ and a large positive number M (such as, we take $\mu = 10^{-10}, \delta = \frac{1}{2^5}$ and $M = 10^{10}$). Choose a positive integer number K and directions $e_1, \ldots, e_K$ (such as, we take $K = 2n$ and $e_i,\ i = 1, \ldots, K$, are the coordinate directions). Choose an initial value q_0 for the parameter q (such as, we take $q_0 = 10$). Let $x_0 \in X$ be a given initial point and let $k := 0$. If $f(x_0) \leq \mu$, then let $x_k^* := x_0$ and go to *Step 6*. Otherwise, let $q := q_0$ and go to *Step 1*.

Step 1. Solve problem (OP) starting from x_k using some local optimization methods. Let x_k^* be a local minimizer. If $\varphi(x_k^*) \leq \mu$, go to *Step 6*; otherwise, set $i := 1$ and take a positive number $\delta_0 > 0$, let

$\delta := \delta_0$.goto *Step 2*.

Step 2. Let $\bar{x}_k^* = x_k^* + \delta e_i$. If $f(\bar{x}_k^*) < f(x_k^*)$, then set $x_{k+1} := \bar{x}_k^*$, $k := k+1$ and go to *Step 1*; otherwise, go to *Step 3*.

Step 3. Construct the following filled function

$$\Psi_{q,x_k^*}(x) = \frac{1}{\|x - x_k^*\|^2 + 1}\left(g_{\frac{f(x_k^*)}{4}}\left(\varphi(x) - \frac{\varphi(x_k^*)}{2}\right)\right) + q\ f_{\frac{f(x_k^*)}{4}}\left(\varphi(x) - \frac{\varphi(x_k^*)}{2}\right), \tag{4.2}$$

where $g_r(t)$ and $f_r(t)$ are defined by (2.2) and (2.1), respectively. Solve the problem

$$\min_{x \in X} \quad \Psi_{q,x_k^*}(x). \tag{4.3}$$

by a local search method starting from the point $\bar{x}_k^*$. Let y_k^* be a local minimizer. If $f(y_k^*) < f(x_k^*)$, then let $x_{k+1} := y_k^*, k := k+1$, goto *Step 1*; otherwise, goto *Step 4*.

Step 4. If $i < K$, then let $i := i+1$, go to *Step 2*; otherwise, go to *Step 5*.

Step 5. If $q < M$ and $r > \mu$, increase q and decrease r; otherwise, go to *Step 6*.

Step 6. Stop. x_k^* is a solution or a μ-approximate solution of (NE).

5. Numerical examples

EX1. Rastrigin $(n = 2)$

$$\min \; f_R(x) = x_1^2 + x_2^2 - \cos(18x_1) - \cos(18x_2) \tag{5.1}$$

$$\text{s.t.} \;\; -2 \le x_i \le 2, \; i = 1, 2.$$

◊ From Figure 5.3, we see that there are many local minima of this problem.

◊ Table 1 gives the numerical results obtained by Algorithm 1 [2] for problem (5.1).

◊ From Table 1, we see that the first local minimizer of problem (5.1) from the first initial point $x_1^0 = (1, 1)^T$ is $x_1^* = (1.0408, 1.0408)$. By the filled function, we find other several initial points:

Table 1: Results for **Rastrigin** by **QFFM**

k	x_k^0	x_k^*	$f(x_k^*)$	$\delta,\ e_i,\ q,\ r$	$\bar{x}_{q,r,x_k^*}$
1	$(1,\ 1)$	$(1.0408, 1.0408)$	0.1798		
				$1/2^4,\ e_1,\ 10^5,\ 1$	$(1.1241, 1.0408)$
2	$(1.1241, 1.0408)$	$(0.3469, 1.0408)$	-0.7890		
				$1/2^4,\ e_1,\ 10^5,\ 1$	$(0.7132, 1.0407)$
3	$(0.7132, 1.0407)$	$(-0.0000, 1.0408)$	-0.9101		
				$1/2^4,\ e_1,\ 10^5,\ 1$	$(0.7515, 1.0405)$
4	$(0.7515, 1.0405)$	$(0.0000, 0.6938)$	-1.5156		
				$1/2^4,\ e_1,\ 10^5,\ 1$	$(0.7512, 0.6936)$
5	$(0.7512, 0.6936)$	$1.0 \times 10^{-6}(0.1434, -0.3560)$	-2.0000		
				for any $e_i, i = 1, \cdots, 2n$ $q \leq 10^{10}$ and $r \geq 10^{-10}$	$\bar{x}_{q,r,x_k^*}$
	$\bar{x}_{q,r,x_k^*}$	x_{k+1}^*	$f(x_{k+1}^*) \geq -2.0000$		

$x_2^0 - x_5^0$, then we obtain other several local minimizers $x_2^* - x_5^*$ of problem (5.1). The x_5^* is the approximate global minimizer of problem (5.1) obtained by **Algorithm 1** in request of the precision $\mu = 10^{-10}$ (since for any e_i, q, r, we can not find better point, the point x_5^* is the global minimizer).

◇ Reference [R1974]: Rastrigin, L., Systems of Extremal Control, Nauka, Moscow, 1974. also obtained the same global minimizer.

EX2. **Two-dimensions Shubert III function** $(n = 2)$ (Shubert, 1972)

$$\begin{aligned} \min \quad f_S(x) = &\left(\sum_{i=1}^{5} i\cos[(i+1)x_1+i]\right) \\ &\cdot \left(\sum_{i=1}^{5} i\cos[(i+1)x_2+i]\right) \\ &+\left[(x_1+1.42513)^2+(x_2+0.80032)^2\right] \\ \text{s.t.} \quad -10 \le x_i \le 10,\ & i=1,2. \end{aligned} \tag{5.2}$$

◇ From Figure 5.4, we see that there are many local minima of this problem (there are about 760 minimums).

Table 2: Results for **Shubert III function** by **QFFM**

k	x_k^0	x_k^*	$f(x_k^*)$	$\delta,\ e_i,,\ q,\ r$	$\bar{x}_{q,r,x_k^*}$
1	$(1,1)$	$(-0.8017, 2.7818)$	-25.0600		
				$1/2^4,\ e_1,\ 10^5,\ 1$	$(-1.4136, 2.2210)$
2	$(-1.4136, 2.2210)$	$(-1.4251, 2.2950)$	-28.0619		
				$1/2^4,\ e_1,\ 10^5,\ 1$	$(-1.4251, -0.8003)$
3	$(-1.4251, -0.8003)$	$(-1.4251, -0.8003)$	-186.7309		
				for any $e_i, i = 1, \cdots, 2n$ $q \leq 10^{10}$ and $r \geq 10^{-10}$	$\bar{x}_{q,r,x_k^*}$
	$\bar{x}_{q,r,x_k^*}$	x_{k+1}^*	$f(x_{k+1}^*) \geq -186.7309$		

◊ Table 2 gives the numerical results obtained by **Algorithm 1** [2] for the **Two-dimensions Shubert III function**.

◊ From Table 2, we see that the first local minimizer of problem (5.2) from the first initial point $x_1^0 = (1,1)^T$ is $x_1^* = (-0.8017, 2.7818)$. By the filled function, we find other two initial points: x_2^0 and x_3^0, then we obtain other two local minimizers x_2^* and x_3^* of problem (5.2). The x_3^* is the global minimizer of problem (5.2) obtained by **Algorithm 1** in request of the precision $\mu = 10^{-10}$, which is the same as the global minimizer given by other references.

EX3. (Test Problem 14.1.1 in [1])

$$\begin{aligned} 4x_1^3 + 4x_1x_2 + 2x_2^2 - 42x_1 - 14 &= 0 \\ 4x_2^3 + 2x_1^2 + 4x_1x_2 - 26x_2 - 22 &= 0 \\ -5 \le x_1, x_2 \le 5 & \end{aligned} \tag{5.3}$$

◊ There are 9 known solutions for this nonlinear equations as shown in [1] (Table 3):

Table 3: Known solutions for Example (5.3)

x_1	-3.7793	-3.0730	-2.8051	-0.2709	-0.1280	0.0867	3.0	3.3852	3.5844
x_2	-3.2832	-0.0814	3.1313	-0.9230	-1.9537	2.8843	2.0	0.0739	-1.8481

◇ Table 4 records the numerical results of solving Example (5.3) by Algorithm 3 [4].

Table 4: Numerical results for Example 5.3

k	x_k	local minimizer x_k^*	$f(x_k^*)$	$F(x_k^*)$
0	$\begin{pmatrix} 1.0000 \\ 3.0000 \end{pmatrix}$	$\begin{pmatrix} -2.8046 \\ 3.1308 \end{pmatrix}$	3.0123×10^{-3}	$\begin{pmatrix} 3.2415 \times 10^{-2} \\ -4.4289 \times 10^{-2} \end{pmatrix}$
1	$\begin{pmatrix} -2.8049 \\ 3.1308 \end{pmatrix}$	$\begin{pmatrix} -2.8051 \\ 3.1313 \end{pmatrix}$	3.3845×10^{-9}	$\begin{pmatrix} 3.0661 \times 10^{-5} \\ -3.0454 \times 10^{-5} \end{pmatrix}$
0	$\begin{pmatrix} -5.0000 \\ -3.0000 \end{pmatrix}$	$\begin{pmatrix} -0.2707 \\ -0.9232 \end{pmatrix}$	3.8212×10^{-5}	$\begin{pmatrix} -5.9582 \times 10^{-3} \\ 1.6470 \times 10^{-3} \end{pmatrix}$
1	$\begin{pmatrix} -0.2708 \\ -0.9232 \end{pmatrix}$	$\begin{pmatrix} -0.2708 \\ -0.9230 \end{pmatrix}$	6.0267×10^{-13}	$\begin{pmatrix} -3.1757 \times 10^{-7} \\ 7.0839 \times 10^{-7} \end{pmatrix}$

◇ It is clear that two solutions are obtained by our algorithm starting from two different initial points.

EX4. (Test Problem 14.1.2 in [1])

$$\begin{aligned}
&x_1x_2 + x_1 - 3x_5 = 0\\
&2x_1x_2 + x_1 + 3R_{10}x_2^2 + x_2x_3^2 + R_7x_2x_3\\
&+R_9x_2x_4 + R_8x_2 - Rx_5 = 0\\
&2x_2x_3^2 + R_7x_2x_3 + 2R_5x_3^2 + R_6x_3 - 8x_5 = 0\\
&R_9x_2x_4 + 2x_4^2 - 4Rx_5 = 0\\
&x_1x_2 + x_1 + R_{10}x_2^2 + x_2x_3^2 + R_7x_2x_3 + x_4^2\\
&+R_9x_2x_4 + R_8x_2 + R_5x_3^2 + R_6x_3 - 1 = 0\\
&0.0001 \le x_i \le 100, i = 1, \ldots, 5,
\end{aligned} \tag{5.4}$$

where

$$\begin{aligned}
&R = 10 && R_5 = 0.193\\
&R_6 = 4.10622 \times 10^{-4} && R_7 = 5.45177 \times 10^{-4}\\
&R_8 = 4.4975 \times 10^{-7} && R_9 = 3.40735 \times 10^{-5}\\
&R_{10} = 9.615 \times 10^{-7}.
\end{aligned}$$

♢ The known solution of Example (5.4) as shown in

[1] is

$$(0.003431, 31.325636, 0.068352, 0.859530, 0.036963)^T.$$

◇ Table 5 records the numerical results of solving Example (5.4) by Algorithm 3 [4].

◇ Here, we find another different approximate solution for Example (5.4).

Table 5: Numerical results for Example (5.4)

k	x_k	local minimizer x_k^*	$f(x_k^*)$	$F(x_k^*)$
0	$\begin{pmatrix} 26.0000 \\ 10.0000\times10^{-5} \\ 6.0000 \\ 10.0000 \\ 10.0000 \end{pmatrix}$	$\begin{pmatrix} 10.0000\times10^{-5} \\ 10.0000\times10^{-5} \\ 2.5706 \\ 4.7886 \\ 0.4109 \end{pmatrix}$	1423.4870	$\begin{pmatrix} -1.2325 \\ -4.1078 \\ -0.7338 \\ 29.4271 \\ 23.2078 \end{pmatrix}$
1	$\begin{pmatrix} 10.0000\times10^{-5} \\ 10.0000\times10^{-5} \\ 2.5706 \\ 4.7888 \\ 0.5119 \end{pmatrix}$	$\begin{pmatrix} 1.1153\times10^{-2} \\ 9.3500 \\ 0.1243 \\ 0.8579 \\ 3.6804\times10^{-2} \end{pmatrix}$	3.3798×10^{-5}	$\begin{pmatrix} 5.0180\times10^{-3} \\ -2.6706\times10^{-3} \\ 1.2161\times10^{-3} \\ 9.2324\times10^{-6} \\ -8.4303\times10^{-5} \end{pmatrix}$

EX4. (Test Problem 14.1.3 in [1])

$$\begin{aligned} 10^4 x_1 x_2 - 1 &= 0 \\ \exp(-x_1) + \exp(-x_2) - 1.001 &= 0 \\ 5.49 \times 10^{-6} \le x_1 \le 4.553 & \\ 2.196 \times 10^{-3} \le x_2 \le 18.21 & \end{aligned} \tag{5.5}$$

- ◇ The known solution of Example (5.5) as shown in [1] is $(1.450 \times 10^{-5}, 6.8933353)$.
- ◇ Table 6 records the numerical results of solving Example (5.5) by Algorithm 3 [4].
- ◇ Another two different solutions for Example (5.5) are obtained by Algorithm 3.

Table 6: Numerical results for Example (5.5)

k	x_k	x_k^*	$f(x_k^*)$	$F(x_k^*)$
0	$\begin{pmatrix} 4.5000 \\ 18.0000 \end{pmatrix}$	$\begin{pmatrix} 5.7220 \times 10^{-6} \\ 16.8750 \end{pmatrix}$	1.1847×10^{-3}	$\begin{pmatrix} -3.4405 \times 10^{-2} \\ -1.0057 \times 10^{-3} \end{pmatrix}$
1	$\begin{pmatrix} 5.7873 \times 10^{-6} \\ 16.8692 \end{pmatrix}$	$\begin{pmatrix} 5.9893 \times 10^{-6} \\ 16.8692 \end{pmatrix}$	1.0810×10^{-4}	$\begin{pmatrix} 1.0348 \times 10^{-2} \\ -1.0060 \times 10^{-3} \end{pmatrix}$
2	$\begin{pmatrix} 5.9474 \times 10^{-6} \\ 16.9709 \end{pmatrix}$	$\begin{pmatrix} 5.9158 \times 10^{-6} \\ 16.8700 \end{pmatrix}$	5.0373×10^{-6}	$\begin{pmatrix} -2.0063 \times 10^{-3} \\ -1.0059 \times 10^{-3} \end{pmatrix}$
3	$\begin{pmatrix} 5.9320 \times 10^{-6} \\ 16.8547 \end{pmatrix}$	$\begin{pmatrix} 5.9896 \times 10^{-6} \\ 16.6939 \end{pmatrix}$	1.0206×10^{-6}	$\begin{pmatrix} -9.2582 \times 10^{-5} \\ -1.0060 \times 10^{-3} \end{pmatrix}$
0	$\begin{pmatrix} 1.0000 \\ 1.0000 \end{pmatrix}$	$\begin{pmatrix} 2.1964 \times 10^{-3} \\ 2.1960 \times 10^{-3} \end{pmatrix}$	1.8951	$\begin{pmatrix} -0.9518 \\ 0.9946 \end{pmatrix}$
1	$\begin{pmatrix} 3.2049 \times 10^{-5} \\ 3.1447 \end{pmatrix}$	$\begin{pmatrix} 3.0075 \times 10^{-5} \\ 3.3296 \end{pmatrix}$	1.2113×10^{-3}	$\begin{pmatrix} 1.3975 \times 10^{-3} \\ 3.4776 \times 10^{-2} \end{pmatrix}$
2	$\begin{pmatrix} 2.2287 \times 10^{-5} \\ 4.4847 \end{pmatrix}$	$\begin{pmatrix} 2.2132 \times 10^{-5} \\ 4.5175 \end{pmatrix}$	9.7922×10^{-5}	$\begin{pmatrix} -1.7902 \times 10^{-4} \\ 9.8939 \times 10^{-3} \end{pmatrix}$
3	$\begin{pmatrix} 1.5358 \times 10^{-5} \\ 6.5332 \end{pmatrix}$	$\begin{pmatrix} 1.4184 \times 10^{-5} \\ 7.0500 \end{pmatrix}$	2.1563×10^{-8}	$\begin{pmatrix} -2.8178 \times 10^{-7} \\ -1.4684 \times 10^{-4} \end{pmatrix}$

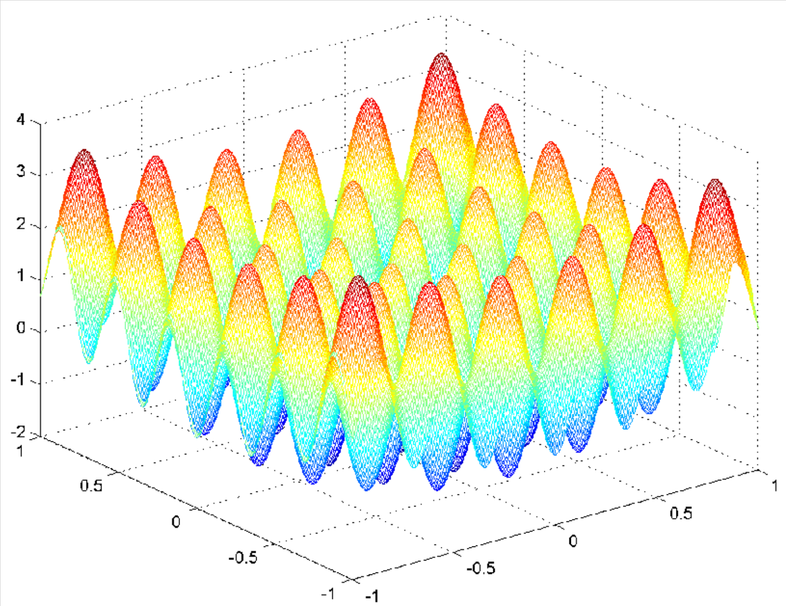

Figure 5.3: The behavior of **Rastrigin**

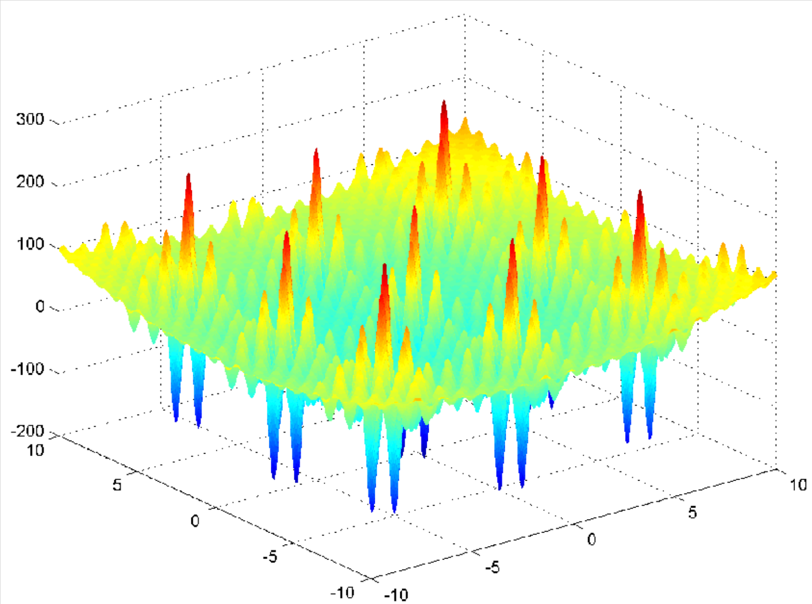

Figure 5.4: The behavior of **two-dimension Shubert III function**

References

[1] C. A. Floudas, P. M. Pardalos, C. S. Adjiman, W. R. Esposito, Z. H. Gumus, S. T. Harding, J. L. Klepeis, C. A. Meyer and C. A. Schweiger, *Handbook of Test Problems in Local and Global Optimization*, Kluwer Academic Publishers, Dordrecht, the Netherlands, 1999.

Thank You!