Classification of Intersection Curves of Two Quadrics

Wenping Wang

The University of Hong Kong

Topics

- Collision detection for ellipsoids, with time parameter t
- Quadric surface intersection curve (QSIC)

Overlap test — (**static objects**)

Determine if two *static* objects overlap or intersect with each other.

Collision detection (CD) — (**moving objects**)

Determine if two *moving* objects come into contact during course of motion

Conventional CD methods

- **Basic idea**: sample time interval of motion and perform overlap test at each sampled time instant
- **Drawback**: collision can be missed between sampled times
- **Speed-up techniques**: frame coherence, bounding volume

Continuous collision detection (CCD)

- no time sampling is necessary
- a univariate equation (or equations) $\Delta(t)$ involving time variable is processed directly.
- the zeros of $\Delta(t)$ tell if there is collision

Separation of Two Ellipses in 2D

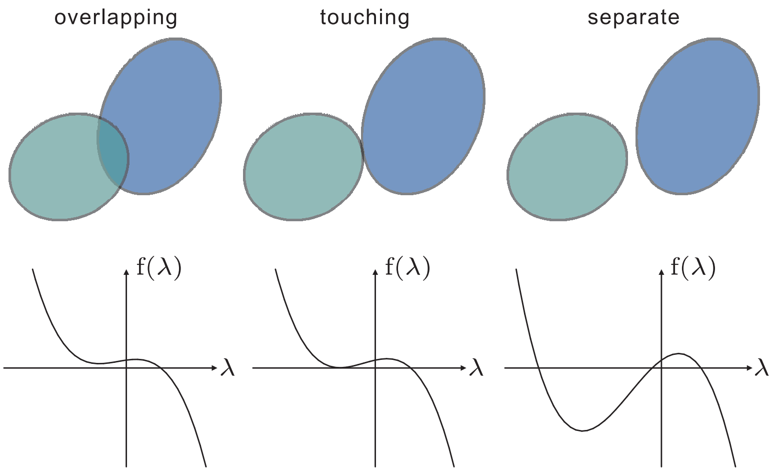
[Choi, W., Liu and Kim, IEEE Tran. Robotics, 2006]

The *characteristic polynomial* of A and B is

$$f(\lambda) \equiv \det(\lambda A - B) = 0.$$

$f(\lambda) = 0$ is *characteristic equation.*

$f(\lambda)$ is cubic, with at least one real zero.



Overlapping $\iff$ $\mathrm{f}(\lambda)$ has no negative root.

Touching externally $\iff$ $\mathrm{f}(\lambda)$ has a double negative root.

Separate $\iff$ $\mathrm{f}(\lambda)$ has two distinct negative roots.

For two moving ellipses $A(t)$ and $B(t)$, write

$$\mathrm{f}(\lambda; t) = \mathrm{g}_3(t)\lambda^3 + \mathrm{g}_2(t)\lambda^2 + \mathrm{g}_1(t)\lambda + \mathrm{g}_0(t).$$

The discriminant of $\mathrm{f}(\lambda; t)$ with respect to λ is

$$\Delta(t) = 18\mathrm{g}_3\mathrm{g}_2\mathrm{g}_1\mathrm{g}_0 - 4\mathrm{g}_2^3\mathrm{g}_0 + \mathrm{g}_2^2\mathrm{g}_1^2 - 4\mathrm{g}_3\mathrm{g}_1^3 - 27\mathrm{g}_3^2\mathrm{g}_0^2.$$

$\mathrm{f}(\lambda; t)$ has a multiple root at t_0 iff $\Delta(t_0) = 0$.

Hence, the collision detection problem is reduced to detecting if $\Delta(t) = 0$ has a real zero in $[0, 1]$.

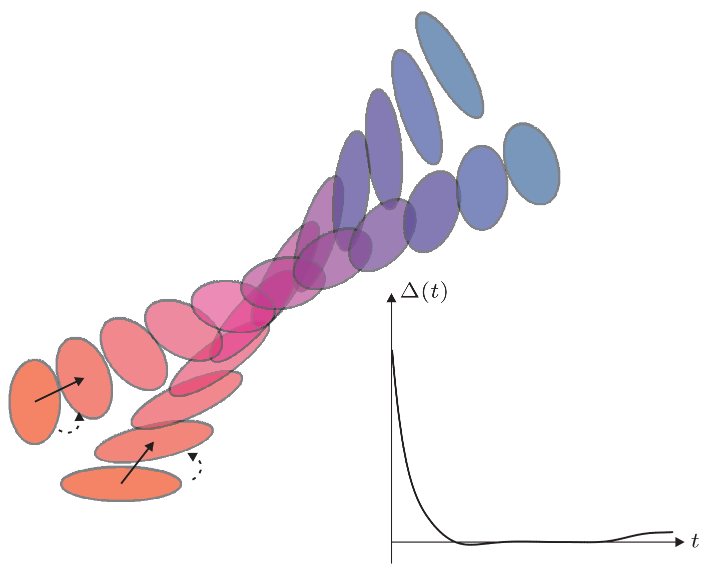
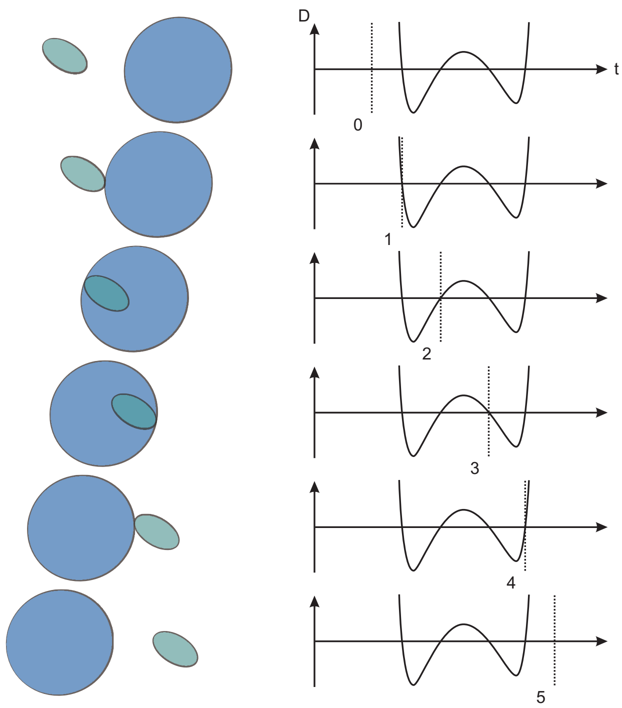
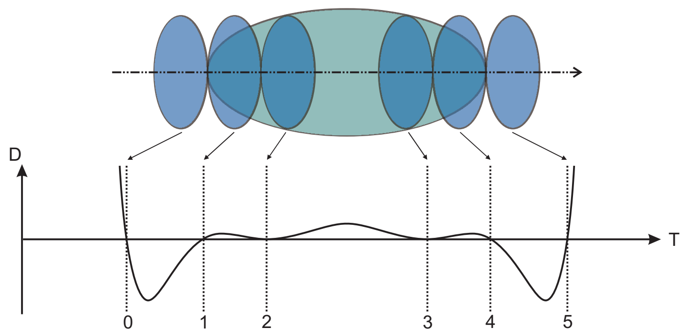


Table 1: Degrees of various entities for rational motions of different degrees. The last row shows the maximum degrees of the entities for a general motion of degree k.

Motion Type	Degree in t: $M(t)$	$g_i(t)$	$\Delta(t)$
Linear Translation	1	$0_{\ (g_0, g_3)},\ 2_{(g_1, g_2)}$	8
General Motion	k	$6k$	$24k$




D
T
0
1
2
3
4
5

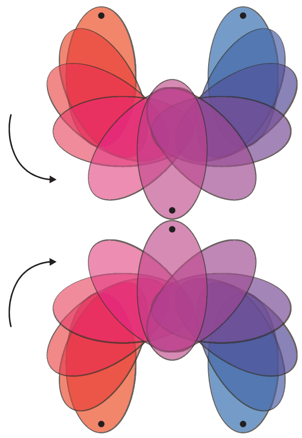
Consider two ellipses $A : \frac{x^2}{5^2} + \frac{y^2}{10^2} = 1$ and $B : \frac{x^2}{5^2} + \frac{y^2}{10^2} = 1$. Two moving elliptic disks $A(t)$ and $B(t)$, $t \in [0, 1]$, are defined by applying to A and B the following motions M_A and M_B:

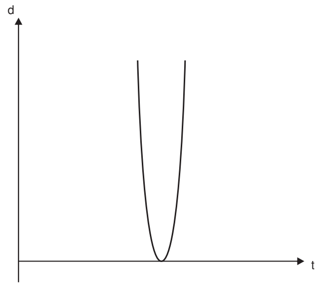
$$M_A = \begin{pmatrix} \begin{matrix} -16t^4 + 32t^3 \\ -16t + 4 \end{matrix} & \begin{matrix} -32t^3 + 48t^2 \\ -16t \end{matrix} & \begin{matrix} -160t^3 - 240t^2 \\ +160t - 40 \end{matrix} \\ \begin{matrix} 32t^3 - 48t^2 \\ +16t \end{matrix} & \begin{matrix} -16t^4 + 32t^3 \\ -16t + 4 \end{matrix} & \begin{matrix} 480t^4 - 960t^3 \\ +880t^2 - 400t + 80 \end{matrix} \\ 0 & 0 & \begin{matrix} 16t^4 - 32t^3 + 32t^2 \\ -16t + 4 \end{matrix} \end{pmatrix},$$

$$M_B = \begin{pmatrix} \begin{matrix} -16t^4 + 32t^3 \\ -16t + 4 \end{matrix} & \begin{matrix} 32t^3 - 48t^2 \\ +16t \end{matrix} & \begin{matrix} 160t^3 - 240t^2 \\ +160t - 40 \end{matrix} \\ \begin{matrix} -32t^3 + 48t^2 \\ -16t \end{matrix} & \begin{matrix} -16t^4 + 32t^3 \\ -16t + 4 \end{matrix} & \begin{matrix} -480t^4 + 960t^3 \\ -880t^2 + 400t - 80 \end{matrix} \\ 0 & 0 & \begin{matrix} 16t^4 - 32t^3 + 32t^2 \\ -16t + 4 \end{matrix} \end{pmatrix}.$$

The characteristic equation is

$$\begin{aligned}
&\mathrm{f}(\lambda; t) = \det(\lambda A(t) - B(t)) \\
&=(-4096t^{24}+49152t^{23}-294912t^{22}+1171456t^{21}-3446784t^{20}+7974912t^{19} \\
&\quad -15048704t^{18}+23721984t^{17}-31756032t^{16}+36517888t^{15}-36360192t^{14} \\
&\quad +31509504t^{13}-23835904t^{12}+15754752t^{11}-9090048t^{10}+4564736t^{9}-1984752t^{8} \\
&\quad +741312t^{7}-235136t^{6}+62304t^{5}-13464t^{4}+2288t^{3}-288t^{2}+24t-1)\lambda^3 \\
&\quad +(-135168t^{24}+1622016t^{23}-11304960t^{22}+55959552t^{21}-206878720t^{20} \\
&\quad +588967936t^{19}-1325514752t^{18}+2409461760t^{17}-3596409600t^{16}+4461631488t^{15} \\
&\quad -4639457280t^{14}+4065807360t^{13}-3011391744t^{12}+1886084608t^{11}-997282816t^{10} \\
&\quad +443501312t^{9}-164884848t^{8}+50819520t^{7}-12842880t^{6}+2624352t^{5}-426168t^{4} \\
&\quad +53808t^{3}-5120t^{2}+344t-13)\lambda^2 \\
&\quad +(135168t^{24}-1622016t^{23}+11304960t^{22}-55959552t^{21}+206878720t^{20} \\
&\quad -588967936t^{19}+1325514752t^{18}-2409461760t^{17}+3596409600t^{16}-4461631488t^{15} \\
&\quad +4639457280t^{14}-4065807360t^{13}+3011391744t^{12}-1886084608t^{11}+997282816t^{10} \\
&\quad -443501312t^{9}+164884848t^{8}-50819520t^{7}+12842880t^{6}-2624352t^{5}+426168t^{4} \\
&\quad -53808t^{3}+5120t^{2}-344t+13)\lambda \\
&\quad +4096t^{24}-49152t^{23}+294912t^{22}-1171456t^{21}+3446784t^{20}-7974912t^{19} \\
&\quad +15048704t^{18}-23721984t^{17}+31756032t^{16}-36517888t^{15}+36360192t^{14} \\
&\quad -31509504t^{13}+23835904t^{12}-15754752t^{11}+9090048t^{10}-4564736t^{9}+1984752t^{8} \\
&\quad -741312t^{7}+235136t^{6}-62304t^{5}+13464t^{4}-2288t^{3}+288t^{2}-24t+1 \\
&=0
\end{aligned}$$





The accuracy of root solving is this case is 10^{-6} with double float precision.

When two elliptic disks assume motions of degree 6, the degree of $\Delta(t)$ is 144.

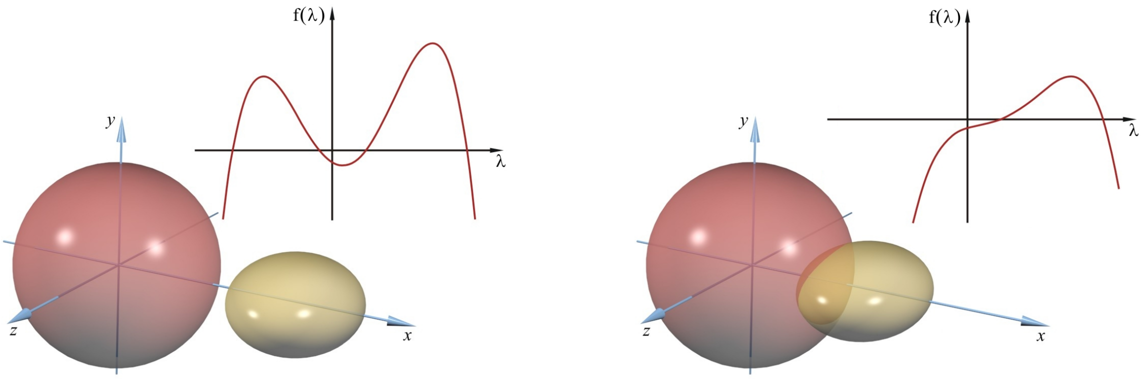
- Detecting collection : 1 ms
- Compute first contact : 5 ms
- Compute all contacts: 7 ms

Separation Test for Ellipsoids

[W., Wang and Kim, CAGD 2001]

For two ellipsoids (A, B), its characteristic polynomial is defined as

$$f(\lambda) = |\lambda A - B|.$$



Two ellipsoids and the corresponding characteristic polynomials.

a) *Two ellipsoids are* separate $\Longleftrightarrow$ $f(\lambda) = 0$ *has two distinct negative roots.*

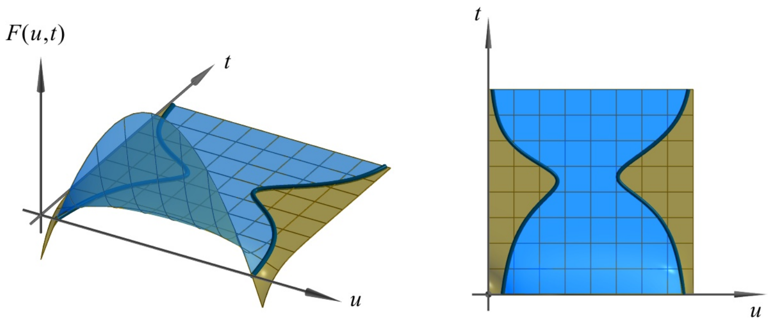
b) Two ellipsoids touch each other externally $\Longleftrightarrow$ $f(\lambda) = 0$ *has a negative double root.*

CD for Ellipsoids

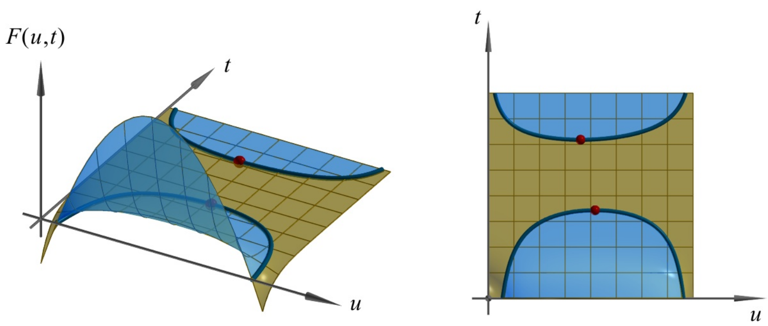
[Choi, Chang, W., Kim, Elber, IEEE TVCG 2005]

Collision is detected by computing the λ-extreme points of the zero set of

$$\mathrm{f}(\lambda; t) \equiv \det(\lambda A(t) - B(t)) = 0.$$



For two collision-free ellipsoids.

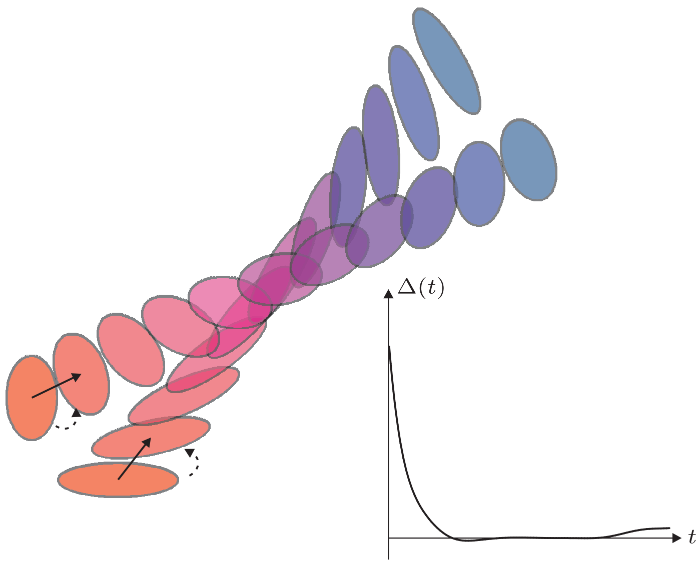


For two colliding ellipsoids.

Further Issues

Question 1.

a) Can collision detection for ellipsoids be done by processing one or more univariate equations of t, like the case of ellipses?



b) Can the separation condition be proved *mechanically*?

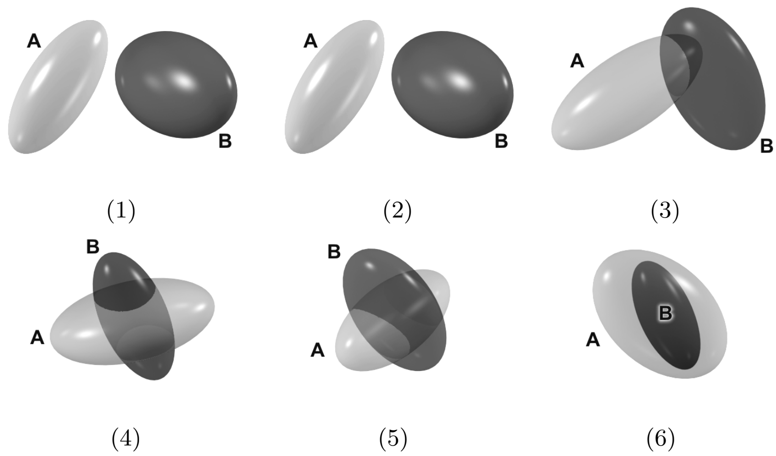


Figure 1: Non-degenerate configurations of two ellipsoids.

Question 2. How to classify all configurations of two ellipsoids?

[W. and Krasauskas, ASM 2004]

(1) $[1|0|\hat{1}|2|3]$ – separated ellipsoids;

(2) $[\hat{1}|2|3]$ – the intersection curve has one component;

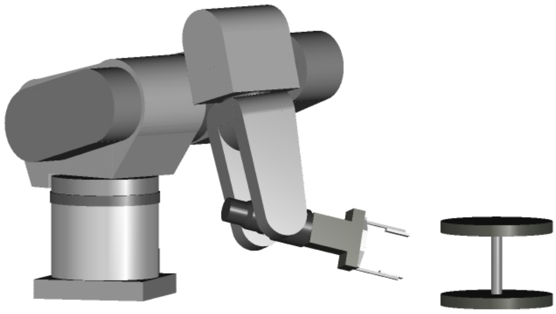
(3) $[\hat{1}|2|1|2|3]$ – the part of $\mathcal{B}$ in the exterior of $\mathcal{A}$ has two components;

(4) $[\hat{1}|2|3|2|3]$ – the part of $\mathcal{A}$ in the exterior of $\mathcal{B}$ has two components;

(5) $[\hat{1}|0|1|2|3]$ – $\mathcal{A}$ contains $\mathcal{B}$;

(6) $[\hat{1}|2|3|4|3]$ – $\mathcal{B}$ contains $\mathcal{A}$.

Question 3. How to test collision detection between quadric surface patches, rather than just complete quadric surfaces (e.g.. ellipsoids)?



Classification of QSIC in $\mathbb{PR}^3$

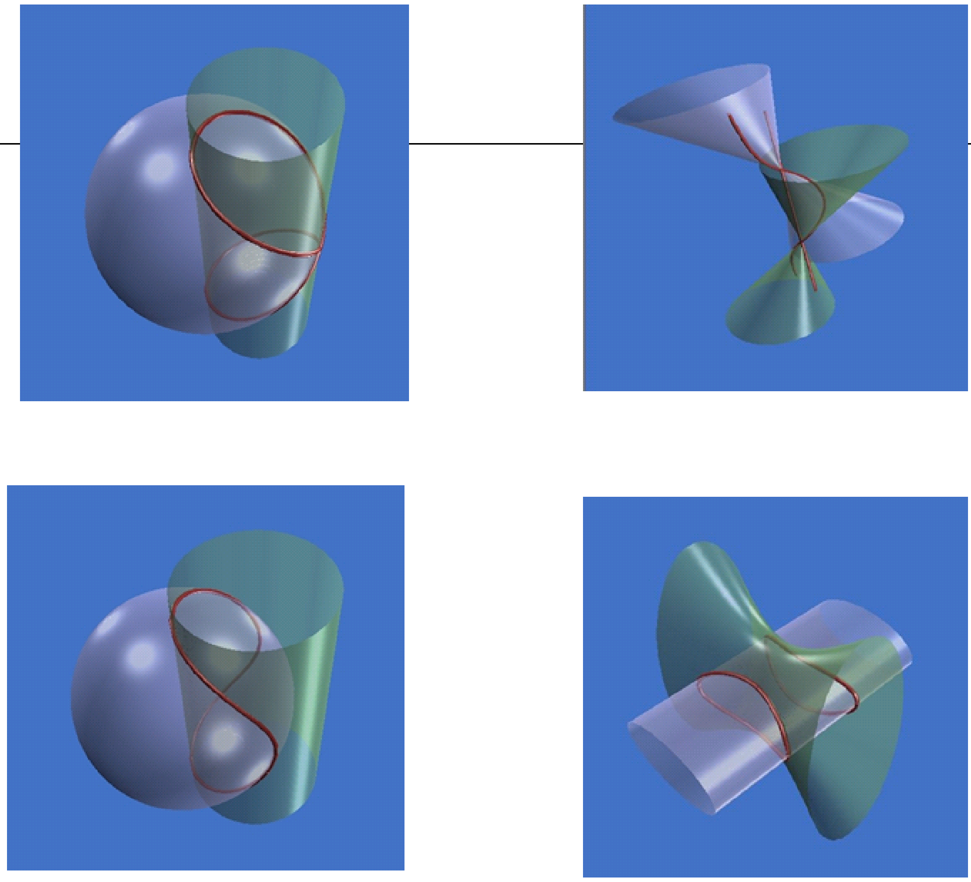


Figure 2: Some examples of QSIC.

Characteristic polynomial

For two quadrics A and B, the characteristic polynomial is

$$f(\lambda) = |\lambda A - B|$$

Proposition: *QSIC is degenerate* $\Longleftrightarrow f(\lambda) = 0$ *has a multiple root.*

A QSIC is degenerate if it has singular points or is reducible.

Previous Works

Most previous works in CAGD on QSIC classification use a procedural approach.

Procedural methods

[Levin, 1976, Comm. ACM]

[Ocken, Schwartz and Sharir, 1987]

[Farouki, Neff and O'Connar, 1989, ACM TOG]

[Wilf and Mannor 1993, CAD]

[Wang, Joe and Goldman, 2002, Graphical Models]

[Wang, Goldman and Tu, 2003, CAGD]

[DuPont et al, 2003, ACM SCG]

Algebraic classification

[Tu, W., Wang, GMP 2002] – non-degenerate QSIC

[Tu et. al. 2005]

[DuPont et. al. 2005]

Classification in $\mathbb{PC}^3$

[Bromwich, 1906]

Characterization of QSIC in $\mathbb{PC}^3$ by Segre characteristics.

- Segre characteristics do not distinguish real and imaginary roots of $f(\lambda) = 0$
- No distinction is made in $\mathbb{PC}^3$ between real and imaginary components of QSIC

E.g., the Segre symbol [1111] means that the four roots of $f(\lambda)$ are distinct and this covers four different types of QSIC in $\mathbb{PR}^3$.

Classification of non-singular QSIC

[Tu, W., Wang, GMP 2002]

Clearly, $f(\lambda)$ *has 0, 2, or 4 real roots, counting multiplicities.*

1) The QSIC has two *non-0-homotopic* components in $\mathbb{PR}^3$ $\Longleftrightarrow$ $f(\lambda) = 0$ has two distinct pairs of complex conjugate roots.

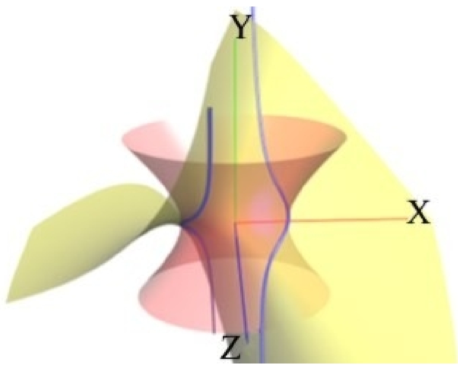


Figure 3: The case of the QSIC has two open components.

2) The QSIC has one (*0-homotopic*) component in $\mathbb{PR}^3 \Longleftrightarrow$ $\mathrm{f}(\lambda) = 0$ has two distinct real roots and a pair of complex conjugate roots.

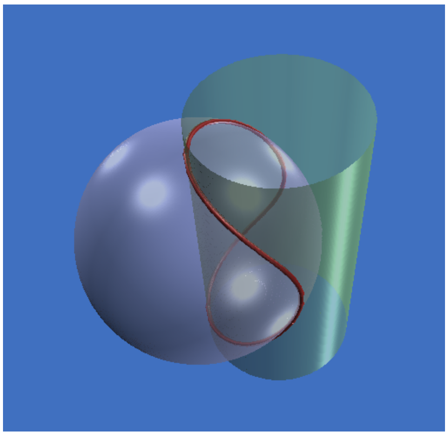


Figure 4: The case of the QSIC has one closed component.

3) The QSIC of $\mathcal{A}$ and $\mathcal{B}$ has either two *0-homotopic* components or no real points in $\mathbb{PR}^3 \Longleftrightarrow \mathrm{f}(\lambda) = 0$ has four distinct real roots.

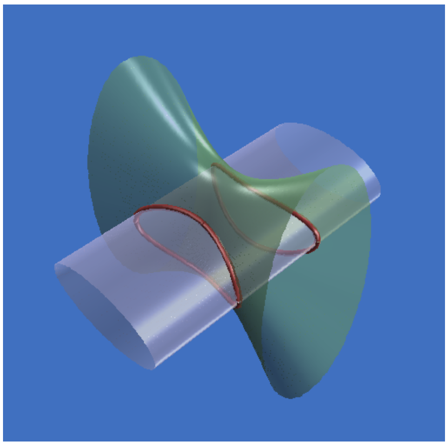


Figure 5: The case of the QSIC has two closed components.

Index function

The *signature* of a real symmetric matrix Q is $(\sigma_+, \sigma_-, \sigma_0)$.

σ_+: # of positive eigenvalues

σ_-: # of negative eigenvalues

σ_0: # of eigenvalues that are zero

$$\sigma_+ + \sigma_- + \sigma_0 = 4$$

The index function of $\lambda A - B$ is $Id(\lambda) = \sigma_+$

Property: $Id(\lambda)$ is constant between any two consecutive real zeros of $f(\lambda)$.

Index sequence

An index sequence is the sequence of indices of a quadric pencil $\lambda A - B$ in consecutive intervals separated by the real zeros of $f(\lambda)$.

$$\langle s_0 \uparrow s_1 \uparrow \ldots \uparrow s_{r-1} \uparrow s_r \rangle,$$

where $\uparrow$ stands for a real root, single or multiple.

Note that

$$s_0 + s_r = \mathrm{rank}(A).$$

Equivalence among index sequences

Any two distinct members in a quadric pencil intersect at the same QSIC – *base curve* of the pencil.

Therefore there is an equivalence relation among all index sequences with respect to projective transformations of λ.

Think of the projective line of λ as a topological circle.

For instance, $\langle 0|1|2|3|4\rangle$ is equivalent one of the following:

a) $\langle 1|2|3|4|3\rangle$; (rotation)

b) $\langle 1|0|1|2|3\rangle$; (rotation)

c) $\langle 4|3|2|1|0\rangle$; (negation or flip)

d) $\langle 3|2|1|0|1\rangle$; (flip of (b))

e) etc.

Degenerate pencils

If $f(\lambda) \equiv 0$, then $(A,\, B)$ form a degenerate pencil.

We shall only consider quadrics pairs for which $f(\lambda)$ does not vanish identically.

Canonical forms of two quadrics

Simultaneous block diagonalization – [Uhlig, 1976]

Two real symmetric matrices A and B, where $|A| \neq 0$, are congruent simultaneously to *Uhlig normal forms*

$$A' = \mathrm{diag}[\varepsilon_1 E_1, \ldots, \varepsilon_k E_k, E_{k+1}, \ldots, E_m]$$

and

$$B' = \mathrm{diag}[\varepsilon_1 E_1 J_1, \ldots, \varepsilon_k E_k J_k, E_{k+1} J_{k+1}, \ldots, E_m J_m]$$

— J_i are Jordan blocks of eigenvalues λ_i of $A^{-1}B$;

— E_i are anti-symmetric unit matrices;

– J_i is of size $2n_i \times 2n_i$ for a complex conjugate root $\lambda_i = a + ib$ of multiplicity n_i;

— $\varepsilon_i = \pm 1$.

Example

Suppose that $f(\lambda) = \det(\lambda A - B)$ has the Segre characteristic [31]. Then the normal forms are

$$A' = \begin{bmatrix} & & \varepsilon_1 & \\ & \varepsilon_1 & & \\ \varepsilon_1 & & & \\ & & & \varepsilon_2 \end{bmatrix}, \quad B' = \begin{bmatrix} & & \varepsilon_1\lambda_1 & \\ & \varepsilon_1\lambda_1 & \varepsilon_1 & \\ \varepsilon_1\lambda_1 & \varepsilon_1 & & \\ & & & \varepsilon_2\lambda_2 \end{bmatrix}$$

Question 1: How does the index change at a real root of $f(\lambda)$?

Question 2: What is the shape of this QSIC?

Answers in this case:

The index sequence is found to be $\langle 1 \wr\wr 2 | 3 \rangle$.

The signature sequence is $(1, (((1,2))), 2, (2,1), 3)$.

The QSIC has a cusp.

Index jump

The index jump $\Delta(\alpha)$ tells how the index function changes across a real root α of $\det(\lambda A - B) = 0$.

$$\Delta(\alpha) = \mathrm{Id}(\alpha_+) - \mathrm{Id}(\alpha_-).$$

If α is a multiple root of $|\lambda I - A^{-1}B| = 0$ associated with several Jordan blocks,

$$\Delta(\alpha) = \sum_{i=1}^{k} \epsilon_i \Delta_i(\alpha).$$

Taking into account the sign $\epsilon_i = \pm$, we can show

i) $\epsilon_i \Delta_i(\alpha) = \pm 1$ if J_i has the size 1×1 or 3×3;

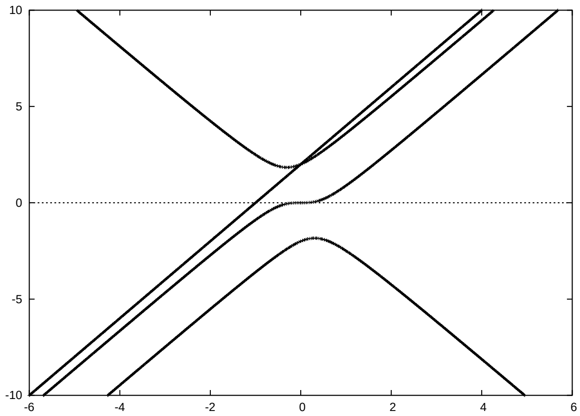
ii) and $\epsilon_i \Delta_i(\alpha) = 0$ if J_i has the size 2×2 or 4×4.

Eigenvalue curve

The eigenvalue curve of two quadrics $(A,\, B)$ is defined as

$$|\lambda A - B - \mu I| = 0.$$

For example, the eigenvalue curve of quadrics $(y^2 + 2\,x\,z + 1,\ 2\,y\,z + 1)$ is shown below.



Proposition 1: The eigenvalue branch $\rho(\lambda)$ of $N_k(\lambda, \rho_0, \varepsilon)$ which vanishes for $\lambda = \rho_0$ is of the form

$$\rho = \varepsilon \nu^k (1 + o(\nu))$$

where $\lambda = \rho_0 + \nu$.

Outline of Analysis

We establish the correspondence between index sequences and QSIC morphologies as follows.

1. Enumerate all Uhlig normal forms of size 4×4;
2. For each normal form, derive its index sequence;
3. For each normal form, determine its QSIC morphology.

Exact classification

We convert each index sequence to a signature sequence, which can easily be computed for any input quadric pair using exact rational arithmetic, so as to determine the type of its QSIC.

E.g., $\langle 1 \wr\wr\wr 2|3\rangle \longrightarrow (1,(((1,2))),2,(2,1),3)$.

Table 2: Classification of nonplanar QSIC in $\mathbb{PR}^3$

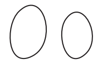
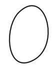
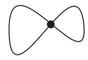
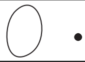
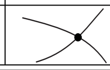
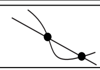
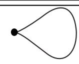
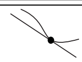
$[\textbf{Segre}]_r$ r = the # of real roots	**Index Sequence**	**Signature Sequence**	**Illustration**	**Representative Quadric Pair**
$[1111]_4$	1 $\langle 1\|2\|1\|2\|3\rangle$	(1,(1,2),2,(1,2),1,(1,2),2,(2,1),3)		$\mathcal{A}:\ x^2+y^2+z^2-w^2=0$ $\mathcal{B}:\ 2x^2+4y^2-w^2=0$
	2 $\langle 0\|1\|2\|3\|4\rangle$	(0,(0,3),1,(1,2),2,(2,1),3,(3,0),4)		$\mathcal{A}:\ x^2+y^2+z^2-w^2=0$ $\mathcal{B}:\ 2x^2+4y^2+3z^2-w^2=0$
$[1111]_2$	3 $\langle 1\|2\|3\rangle$	(1,(1,2),2,(2,1),3)		$\mathcal{A}:\ 2xy+z^2+w^2=0$ $\mathcal{B}:\ -x^2+y^2+z^2+2w^2=0$
$[1111]_0$	4 $\langle 2\rangle$	(2)		$\mathcal{A}:\ xy+zw=0$ $\mathcal{B}:\ -x^2+y^2-2z^2+zw+2w^2=0$
$[211]_3$	5 $\langle 2\wr_- 2\|3\|2\rangle$ $\langle 2\wr_+ 2\|3\|2\rangle$	(2,((2,1)),2,(2,1),3,(2,1),2) (2,((1,2)),2,(2,1),3,(2,1),2)		$\mathcal{A}:\ x^2-y^2+z^2+4yw=0$ $\mathcal{B}:\ -3x^2+y^2+z^2=0$
	6 $\langle 1\wr_- 1\|2\|3\rangle$	(1,((1,2)),1,(1,2),2,(2,1),3)		$\mathcal{A}:\ -x^2-z^2+2yw=0$ $\mathcal{B}:\ -3x^2+y^2-z^2=0$
	7 $\langle 1\wr_+ 1\|2\|3\rangle$	(1,((0,3)),1,(1,2),2,(2,1),3)		$\mathcal{A}:\ x^2+z^2+2yw=0$ $\mathcal{B}:\ 3x^2+y^2+z^2=0$
$[211]_1$	8 $\langle 2\wr_- 2\rangle$	(2,((2,1)),2)		$\mathcal{A}:\ xy+zw=0$ $\mathcal{B}:\ 2xy+y^2-z^2+w^2=0$
$[22]_2$	9 $\langle 2\wr_+ 2\wr_- 2\rangle$ $\langle 2\wr_+ 2\wr_+ 2\rangle$	(2,((2,1)),2,((1,2)),2) (2,((2,1)),2,((2,1)),2)		$\mathcal{A}:\ xy+zw=0$ $\mathcal{B}:\ y^2+2zw+w^2=0$
$[22]_0$	10 $\langle 2\rangle$	(2)		$\mathcal{A}:\ xw+yz=0$ $\mathcal{B}:\ xz-yw=0$
$[31]_2$	11 $\langle 1\wr\wr_+ 2\|3\rangle$	(1,(((1,2))),2,(2,1),3)		$\mathcal{A}:\ y^2+2xz+w^2=0$ $\mathcal{B}:\ 2yz+w^2=0$
$[4]_1$	12 $\langle 2\wr\wr\wr_- 2\rangle$	(2,((((2,1)))),2)		$\mathcal{A}:\ xw+yz=0$ $\mathcal{B}:\ z^2+2yw=0$

Table 3: Classification of planar QSIC in $\mathbb{PR}^3$ - Part I

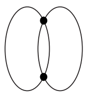
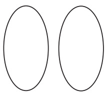
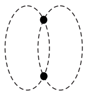
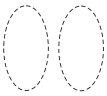
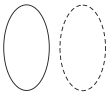
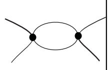
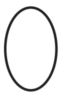
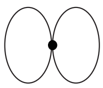
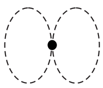
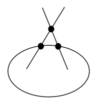
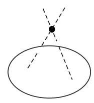
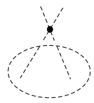
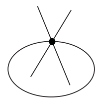
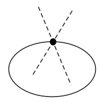
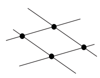
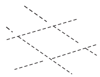
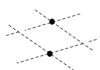
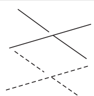
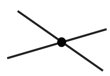
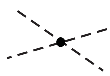
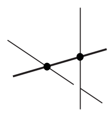
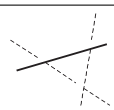
$[\textbf{Segre}]_r$ r = the # of real roots	**Index Sequence**	**Signature Sequence**	**Illus- tration**	**Representative Quadric Pair**
$[(11)11]_3$	13 $\langle 2\|\|2\|1\|2\rangle$	(2,((1,1)),2,(1,2),1,(1,2),2)		$\mathcal{A}:\ x^2-y^2+z^2-w^2=0$ $\mathcal{B}:\ x^2-2y^2=0$
	14 $\langle 1\|\|3\|2\|3\rangle$	(1,((1,1)),3,(2,1),2,(2,1),3)		$\mathcal{A}:\ -x^2+y^2+z^2+w^2=0$ $\mathcal{B}:\ -x^2+2y^2=0$
	15 $\langle 1\|\|1\|2\|3\rangle$	(1,((0,2)),1,(1,2),2,(2,1),3)		$\mathcal{A}:\ x^2+y^2+z^2-w^2=0$ $\mathcal{B}:\ x^2+2y^2=0$
	16 $\langle 0\|\|2\|3\|4\rangle$ $\langle 1\|\|3\|4\|3\rangle$	(0,((0,2)),2,(2,1),3,(3,0),4) (1,((1,1)),3,(3,0),4,(3,0),3)		$\mathcal{A}:\ x^2+y^2-z^2-w^2=0$ $\mathcal{B}:\ x^2+2y^2=0$
$[(11)11]_1$	17 $\langle 1\|\|3\rangle$	(1,((1,1)),3)		$\mathcal{A}:\ x^2+y^2 2zw=0$ $\mathcal{B}:\ -z^2+w^2+2zw=0$
	18 $\langle 2\|\|2\rangle$	(2,((1,1)),2)		$\mathcal{A}:\ x^2-y^2 2zw=0$ $\mathcal{B}:\ -z^2+w^2+2zw=0$
$[(111)1]_2$	19 $\langle 1\|\|\|2\|3\rangle$	(1,(((0,1))),2,(2,1),3)		$\mathcal{A}:\ y^2+z^2-w^2=0$ $\mathcal{B}:\ x^2=0$
	20 $\langle 0\|\|\|3\|4\rangle$	(0,(((0,1))),3,(3,0),4)		$\mathcal{A}:\ y^2+z^2+w^2=0$ $\mathcal{B}:\ x^2=0$
$[(21)1]_2$	21 $\langle 1\wr\wr_-\|2\|3\rangle$	(1,(((1,1))),2,(2,1),3)		$\mathcal{A}:\ y^2-z^2+2zw=0$ $\mathcal{B}:\ -x^2+z^2=0$
	22 $\langle 1\wr\wr_+\|2\|3\rangle$	(1,(((0,2))),2,(2,1),3)		$\mathcal{A}:\ y^2-z^2+2zw=0$ $\mathcal{B}:\ x^2+z^2=0$

Table 4: Classification of planar QSIC in $\mathbb{PR}^3$ - Part II

$[\mathbf{Segre}]_r$ r = the # of real roots	**Index Sequence**	**Signature Sequence**	**Illus- tration**	**Representative Quadric Pair**
	23 $\langle 2\wr\wr_- 2 \| \| 2\rangle$	(2,((1,1)),2,((2,1)),2)		$\mathcal{A}:\ 2xy - y^2 = 0$ $\mathcal{B}:\ y^2 + z^2 - w^2 = 0$
$[2(11)]_2$	24 $\langle 1\wr\wr_- 1 \| \| 3\rangle$	(1,((1,2)),1,((1,1)),3)		$\mathcal{A}:\ 2xy - y^2 = 0$ $\mathcal{B}:\ y^2 - z^2 - w^2 = 0$
	25 $\langle 1\wr\wr_+ 1 \| \| 3\rangle$	(1,((0,3)),1,((1,1)),3)		$\mathcal{A}:\ 2xy - y^2 = 0$ $\mathcal{B}:\ y^2 + z^2 + w^2 = 0$
$[(31)]_1$	26 $\langle 2\wr\wr\wr \| 2\rangle$	(2,((((1,1)))),2)		$\mathcal{A}:\ y^2 + 2xz - w^2 = 0$ $\mathcal{B}:\ yz = 0$
	27 $\langle 1\wr\wr\wr \| 3\rangle$	(1,((((1,1)))),3)		$\mathcal{A}:\ y^2 + 2xz + w^2 = 0$ $\mathcal{B}:\ yz = 0$
	28 $\langle 2 \| \| 2 \| \| 2\rangle$	(2,((1,1)),2,((1,1)),2)		$\mathcal{A}:\ x^2 - y^2 = 0$ $\mathcal{B}:\ z^2 - w^2 = 0$
$[(11)(11)]_2$	29 $\langle 0 \| \| 2 \| \| 4\rangle$	(0,((0,2)),2,((2,0)),4)		$\mathcal{A}:\ x^2 + y^2 = 0$ $\mathcal{B}:\ z^2 + w^2 = 0$
	30 $\langle 1 \| \| 1 \| \| 3\rangle$	(1,((0,2)),1,((1,1)),3)		$\mathcal{A}:\ x^2 + y^2 = 0$ $\mathcal{B}:\ z^2 - w^2 = 0$
$[(11)(11)]_0$	31 $\langle 2\rangle$	(2)		$\mathcal{A}:\ xy + zw = 0$ $\mathcal{B}:\ -x^2 + y^2 - z^2 + w^2 = 0$
$[(211)]_1$	32 $\langle 2\wr\wr \| \| 2\rangle$	(2,((((1,0)))),2)		$\mathcal{A}:\ x^2 - y^2 + 2zw = 0$ $\mathcal{B}:\ z^2 = 0$
	33 $\langle 1\wr\wr \| \| 3\rangle$	(1,((((1,0)))),3)		$\mathcal{A}:\ x^2 + y^2 + 2zw = 0$ $\mathcal{B}:\ z^2 = 0$
$[(22)]_1$	34 $\langle 2\widehat{\wr\wr}_+\widehat{\wr\wr}_- 2\rangle$	(2,((((1,1)))),2)		$\mathcal{A}:\ xy - zw = 0$ $\mathcal{B}:\ y^2 - w^2 = 0$
	35 $\langle 2\widehat{\wr\wr}_+\widehat{\wr\wr}_+ 2\rangle$	(2,((((0,2)))),2)		$\mathcal{A}:\ xy + zw = 0$ $\mathcal{B}:\ y^2 + w^2 = 0$

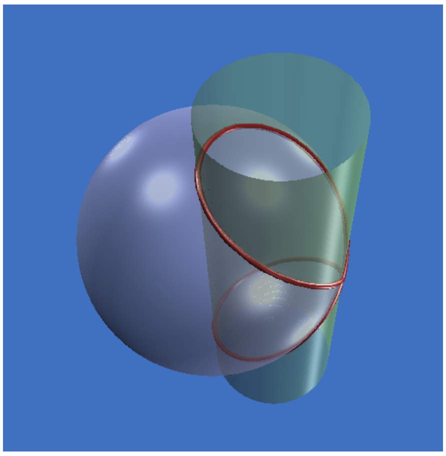


Figure 6: The case of the QSIC having one closed component with a crunode. $\langle 2 \wr 2|3|2 \rangle$

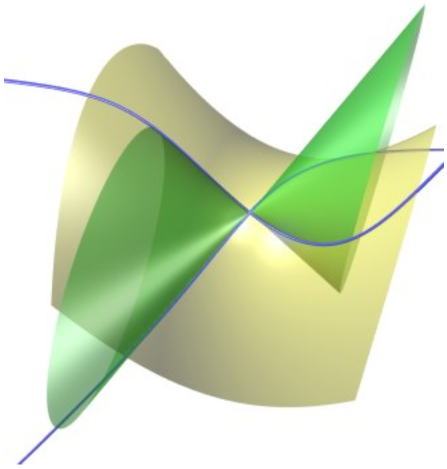
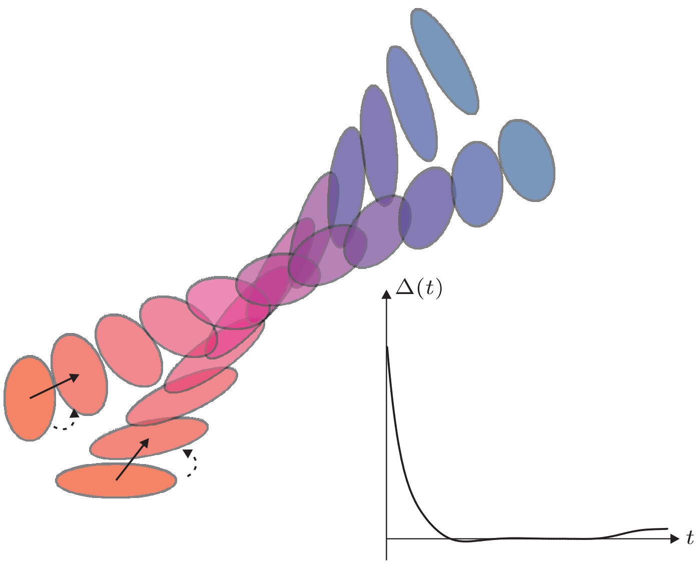


Figure 7: The case of the QSIC having one open component with a crunode. $\langle 2 \wr 2 \rangle$

Discussions

Question 1.

a) Can collision detection for ellipsoids be done by processing one or more univariate equations of t, like the case of ellipses?



b) Can the separation condition be proved *mechanically*?

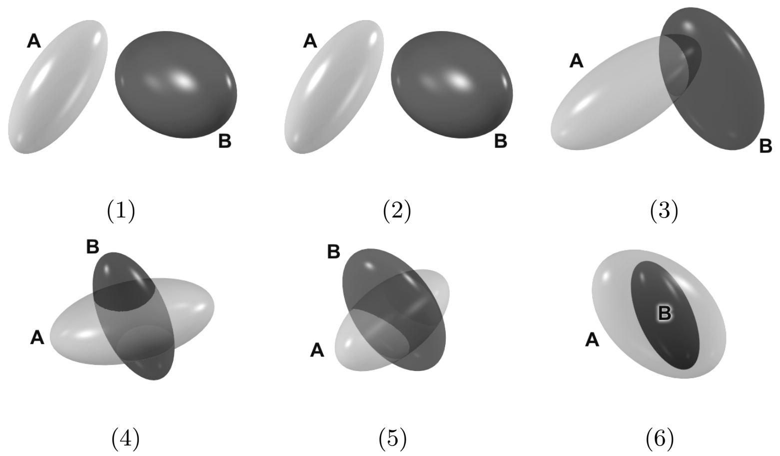


Figure 8: Non-degenerate configurations of two ellipsoids.

Question 2.

a) How to algebraically classify all configurations of two ellipsoids?
(See the full list of 24 cases.)

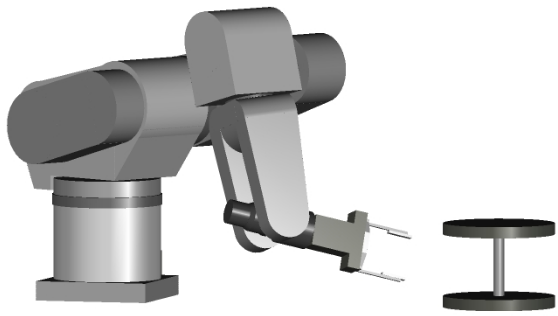
Non-degenerate cases [W. and Krasauskas, ASM 2004]

b) How to generate explicit expressions for these conditions?
Gonzalez has done it for ellipses.

c) What is the state-transition graph of all these 24 configurations of ellipsoids? What is the degeneracy dimension of each configuration.

d) The same question (c) for the 35 types of QSIC.

Question 3. How to test collision detection between quadric surface patches, rather than just complete quadric surfaces (e.g.. ellipsoids)?



CD for composite quadric models (CGM)

A CQM (composite quadric model), or its boundary, is semi-algebraic, making it difficult to take an algebraic approach directly.

We propose a two-step method:

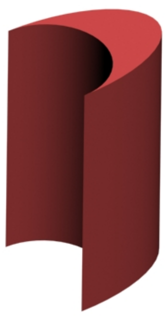
1. Compute candidate contacts using an algebraic method;

2. Verify each candidate contact.

Boundary elements of CQM

- *face* (F): a quadric or a planar surface patch
- *edge* (E): a curve segment or a line segment where two faces meet.
- *vertex* (V): a point where more than two faces meet

Extended boundary element: the complete surface or curve containing a boundary face or edge.

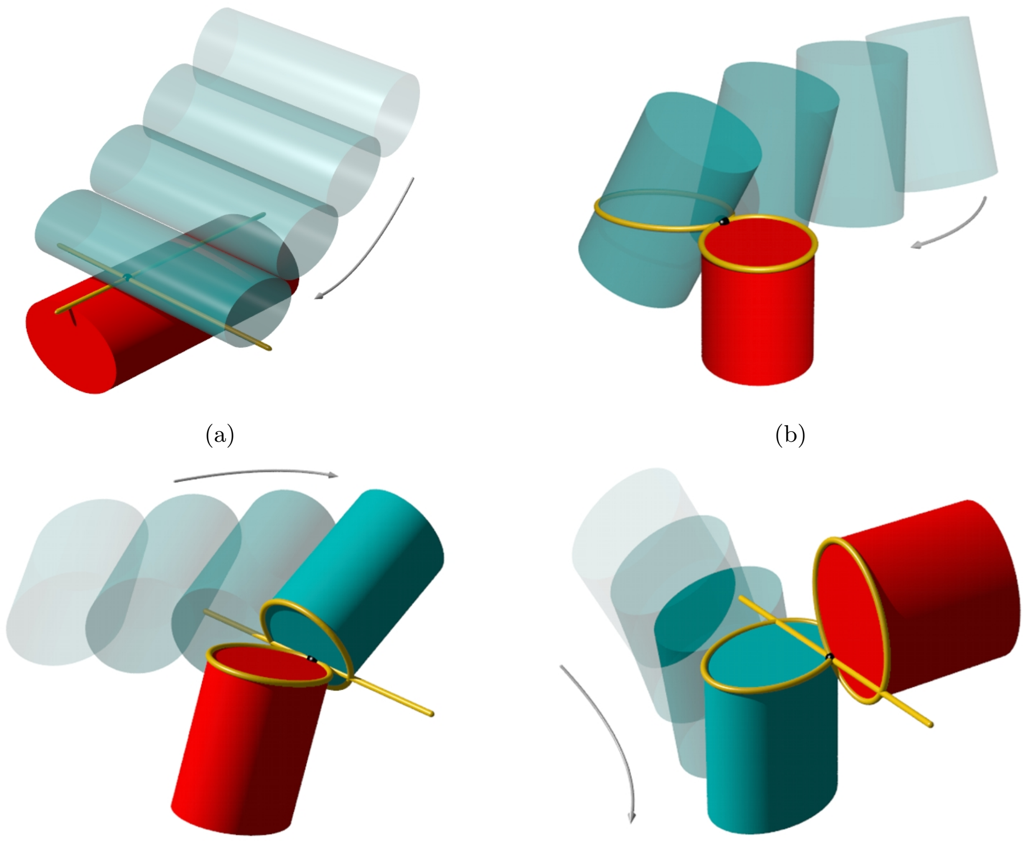


Contact classification

A contact can be in one of the following four types:

(F, F), (F, E), (F, V), (E, E)

Remark: The types (E,V) and (V, V) are encompassed by the above cases.



(a) (b) (c) (d)

Computing candidate contact

A candidate contact occurs between two *extended* boundary elements. It is computed using an algebraic method.

For example, consider the CCD of two capped cylinders.

We need to consider 17 pairs of extended boundary elements.

Each case is reduced to CCD of two cylinders in 3D, two conics in 2D, or two intervals in 1D.

Contact verification

If a candidate point p is detected at time t_0, we need to check if p is on the boundaries of two input CQMs $Q_A(t_0)$ and $Q_B(t_0)$.

Suppose a CQM is a CSG (constructive solid geometry) object, which is obtained by Boolean operations on primitive defined by planes and quadrics.

Then at t_0 it is easy to check if p is a boundary point.

CCD for general CQMs

Type (F,E)

For two general CQMs, an edge may be a general QSIC curve.

Then a contact of type (F, E) occurs when a quadric A is touched by the intersection curve of two quadrics B and C.

Or, detect if three quadrics have a common point in 3D.

This leads to the study of the quadric net

$$\alpha A + \beta B + \gamma C$$

Consider the plane quartic curve

$$H(\alpha, \beta, \gamma) \equiv \det(\alpha A + \beta B + \gamma C) = 0$$

Conjecture: the contact (F, E) occurs $\Longleftrightarrow$ the curve $H = 0$ has a singular point.

Type (E,E)

Then a contact of type (E, E) occurs when the intersection of a pair of quadrics A and B is touched by the intersection curve of a pair of quadrics of C and D.

Or, detect if four quadrics have a common point in 3D.

This leads to the study of the quadric net

$$\alpha A + \beta B + \gamma C + \delta D$$

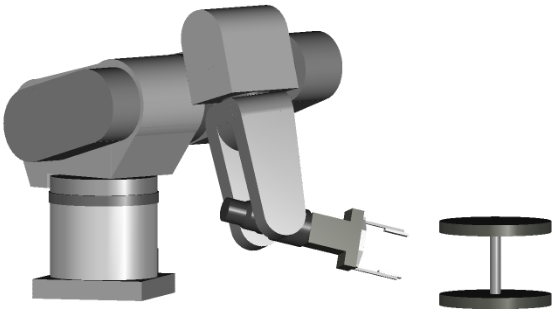
Consider the quartic surface

$$H(\alpha, \beta, \gamma, \delta) \equiv \det(\alpha A + \beta B + \gamma C + \delta D) = 0$$

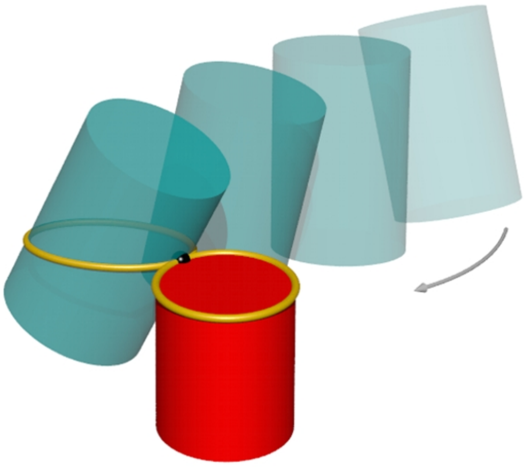
.

Conjecture: the contact (E, E) occurs $\Longleftrightarrow$ the surface $H = 0$ has a singular point.

Special case: all edges are conics

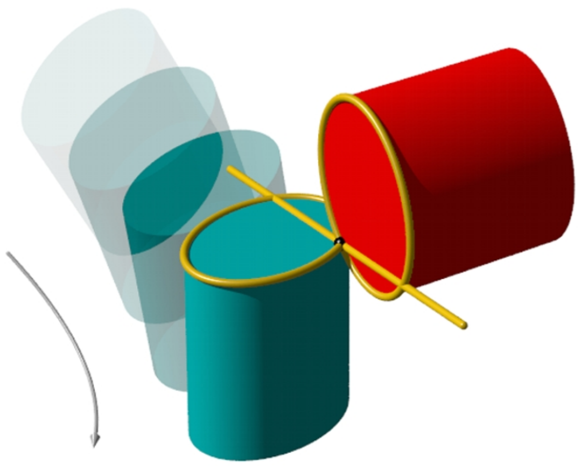


Type (F,E) contact is reduced to the 2D CD problem of two conics.



Type (E,E)

If both edge elements are planar, this can be reduced to a 1D CCD problem of two intervals.



Question 4: Classification of configurations of three ellipses/ellipsoids.

The cases of three ellipses A, B, C leads to the *conic net*

$$\alpha A + \beta B + \gamma C$$

How is each configuration of the triple of ellipses related to the morphology of the plane cubic curve

$$H(\alpha, \beta, \gamma) \equiv \det(\alpha A + \beta B + \gamma C) = 0$$

?

Question 5: Extension beyond quadric surfaces?

High degree surfaces beyond quadrics are widely used in free-form surface modeling. Some special cases: parametric cubic surfaces (degree 9), cyclides (degree 4).

Thank you!