Pillwein's Proof of Schöberl's Conjecture

Manuel Kauers

currently at a INRIA-Rocquencourt usually at SRISC-Linz



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... if we have a fast computer.

▶ Problem 11251 (Marian Tetiva; vol. 113(10), 2006, p. 847): Let a, b, c be positive real numbers, two of which are ≤ 1, satisfying ab + ac + bc = 3. Show that

$$\frac{1}{(a+b)^2} + \frac{1}{(a+c)^2} + \frac{1}{(b+c)^2} - \frac{3}{4} \ge \frac{3(a-1)(b-1)(c-1)}{2(a+b)(a+c)(b+c)}$$

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Problem 11301 (Finbarr Holland; vol. 114(10), 2007, p. 547): Find the least number M such that for all a, b, c,

$$|ab(a^{2} - b^{2}) + bc(b^{2} - c^{2}) + ca(c^{2} - a^{2})| \le M(a^{2} + b^{2} + c^{2})^{2}.$$

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▶ Problem 11251:

$$\begin{aligned} \forall a, b, c : \left(a > 0 \land 1 \ge b > 0 \land 1 \ge c > 0 \land ab + ac + bc = 3 \\ \Rightarrow \frac{1}{(a+b)^2} + \frac{1}{(a+c)^2} + \frac{1}{(b+c)^2} - \frac{3}{4} \ge \frac{3(a-1)(b-1)(c-1)}{2(a+b)(a+c)(b+c)} \end{aligned} \end{aligned}$$

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One such algorithm is due to Collins (Cylindrical Algebraic Decomposition, *CAD*, 1975).

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Polynomial inequalities can be proven by CAD without thinking.

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The rest of this talk is about inequalities that can be proven by CAD with thinking only.

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$$\forall n \in \mathbb{N} \ \forall x \ge -1 : (x+1)^n - (1+nx) \ge 0.$$

 $\forall \ n\in \mathbb{N} \ \forall \ x\geq -1: (x+1)^n-(1+nx)\geq 0.$ What exactly does $(x+1)^n-(1+nx)$ mean?

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- View Bernoulli's inequality as a sequence of polynomial inequalities.





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- Exploit the recurrence $f_{n+1}(x) = (x+1)f_n(x) + nx^2$
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- The resulting formula is indeed true.

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- This completes the proof.

Message:

We may use CAD to construct an induction proof for the positivity of a quantity satisfying a recurrence.

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P₅(x) = ⁶³/₈x⁵ - ³⁵/₄x³ + ¹⁵/₈x
P₆(x) = ²³¹/₁₆x⁶ - ³¹⁵/₁₆x⁴ + ¹⁰⁵/₁₆x² - ⁵/₁₆



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Legendre Polynomials: Turan's Inequality

Here is an inequality about $P_n(x)$ that can be shown with CAD:

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$$\forall n \in \mathbb{N} \ \forall x : -1 \le x \le 1 \Rightarrow \underbrace{P_{n+1}^2(x) - P_n(x)P_{n+2}(x)}_{=:\Delta_n(x)} \ge 0$$



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- For specific n, it is just a polynomial inequality.
- For general n, it is not easy. (Try it.)

A proof for general $n\ {\rm can}\ {\rm be}\ {\rm obtained}\ {\rm in}\ {\rm the}\ {\rm same}\ {\rm way}\ {\rm as}\ {\rm for}\ {\rm Bernoulli's}\ {\rm inequality}.$

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Use the recurrence for $P_n(x)$ to obtain

$$\Delta_n(x) = \frac{(n+1)}{n+2} P_n(x)^2 - \frac{2n+3}{n+2} x P_{n+1}(x) P_n(x) + P_{n+1}(x)^2$$

$$\Delta_{n+1}(x) = \frac{(n+1)^2}{(n+2)^2} P_n(x)^2 - \frac{(n+1)(2n^2+9n+8)x}{(n+2)^2(n+3)} P_{n+1}(x) P_n(x)$$

$$+ \frac{(n+2)^3 - (2n+3)x^2}{(n+2)^2(n+3)} P_{n+1}(x)^2$$

Here is an inequality about $P_n(x)$ that can be shown with CAD:

$$\forall n \in \mathbb{N} \ \forall x : -1 \le x \le 1 \Rightarrow P_{n+1}^2(x) - P_n(x)P_{n+2}(x) \ge 0$$

Relaxing $P_n(x)$ to y, and $P_{n+1}(x)$ to z, and $n\in\mathbb{N}$ to $n\geq 0$ leads to the formula

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which is indeed *true*. This proves the induction step.

The induction base $\Delta_0(x) \ge 0$ is trivial. This completes the proof.

Message:

A "deep" special function inequality may be just an immediate consequence of a polynomial inequality.

Turan's inequality

 $\Delta_n(x) = P_{n+1}(x)^2 - P_n(x)P_{n+2}(x) \ge 0$



$$\Delta_n(x) = P_{n+1}(x)^2 - P_n(x)P_{n+2}(x) \ge \alpha_n(1-x^2)$$



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Can we show this also by induction?



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A Tarski formula encoding the induction step would be...

$$\Delta_n(x) = P_{n+1}(x)^2 - P_n(x)P_{n+2}(x) \ge \alpha_n(1-x^2)$$

$$\begin{split} \forall n, x, y, z, a, b : \left(n \geq 0 \land x^2 \leq 1 \land \frac{n+1}{n+2}y^2 - \frac{2n+3}{n+2}xyz + z^2 \geq a(1-x^2) \\ \land \frac{(n+1)^2}{(n+2)^2}y^2 - \frac{(n+1)(2n^2+9n+8)x}{(n+2)^2(n+3)}yz + \frac{(n+2)^3-(2n+3)x^2}{(n+2)^2(n+3)}z^2 \geq b(1-x^2) \right) \\ \Rightarrow \left(\frac{(n+1)^2((n+3)^3-(2n+5)x^2)}{(n+4)(n+3)^2(n+2)^2}y^2 + \frac{(n+1)(2(2n+3)(2n+5)x^2-(2n^4+21n^3+83n^2+142n+86))}{(n+4)(n+3)^2(n+2)^2}xyz + \frac{((n+4)(n+2)^4-(2n+3)^2(2n+5)x^4+(n+1)(2n+3)(2n+5)x^2)}{(n+4)(n+3)(n+2)}z^2 \\ \geq \frac{(n+1)(n+2)}{(n+3)(n+4)}a + \frac{(2n+5)}{(n+3)(n+4)}b \right). \end{split}$$

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Unfortunately, this is *false*.
Turan's inequality can be improved to

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Unfortunately, this is *false*. We must be more careful.

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Observations:

• By symmetry, it suffices to consider $x \ge 0$.

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Observations:

- By symmetry, it suffices to consider $x \ge 0$.
- For x = 0 there is nothing to show.
- For x > 0, it suffices to show that $\Delta_n(x)/(1-x^2)$ is *increasing*.

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A positivity proof for the latter expression by CAD and induction on \boldsymbol{n} succeeds.

Message:

A special function inequality may require some non-obvious manipulation before an induction proof via CAD succeeds.

 In the higher order finite element method (FEM), solutions of PDEs are locally approximated by polynomials.



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- ► Some basis polynomials lead to better numerical performance than the standard basis 1, x, x², x³,



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 for all q with $\deg q \le n$.

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$$\int_{-1}^{1} f_n(x)q(x)dx = q(0)$$
 for all q with deg $q \le n$
▶ $\int_{-1}^{1} |f_n(x)|dx \le C$ for some constant C .

► The Legendre kernel polynomials

$$k_n(x,y) := \frac{n+1}{2(x-y)} (P_{n+1}(x)P_n(y) - P_{n+1}(y)P_n(x))$$

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- So $f_n(x) := k_n(x, 0)$ satisfies the first condition.
- But not the second.

Schöberl next considered the "gliding averages"

$$f_n(x) := \frac{1}{n+1} \sum_{i=n}^{2n} k_i(x,0).$$

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He could show that this family does the job if and only if...

$$\sum_{k=0}^{n} (4k+1)(2n-2k+1)P_{2k}(0)P_{2k}(x) \ge 0$$

for all $x \in [-1, 1]$ and all $n \in \mathbb{N}$.

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Hence was born the Schöberl conjecture.

Consider

$$S_n(x) := \sum_{k=0}^n (4k+1)(2n-2k+1)P_{2k}(0)P_{2k}(x)$$



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for $n = 0, 1, \dots, 20$. The conjecture seems to be true.



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- For specific n ∈ N, it can be shown without thinking.

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- The conjecture seems to be true.
- For specific n ∈ N, it can be shown without thinking.
- It can be also be shown for x = −1, x = 0, x = +1.
- But a proof for general x, n could not be found for some years.

Message:

Special function inequalities arise in real world applications.

$$\sum_{k=0}^{n} P_k(x) \ge 0 \quad (x \in [-1, 1], n \in \mathbb{N})$$

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where $P_k^{(\alpha,\beta)}(x)$ refers to the Jacobi polynomials. As $P_k(x) = P_k^{(0,0)}(x)$, it includes Fejer's inequality.

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- Now back to Schöberl's conjecture...

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- Proving estimates for each part by hand
- Combining the estimates for both components with CAD and induction

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What is the reason for this gap?

$$S_n^{\alpha}(x) := \sum_{k=0}^n (2\alpha + 4k + 1)(2n - 2k + 1) \frac{\binom{2k+2\alpha}{\alpha}}{4^{\alpha}\binom{2k+\alpha}{\alpha}} P_{2k}^{(\alpha,\alpha)}(0) P_{2k}^{(\alpha,\alpha)}(x)$$

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Conjecture: $S_n^{\alpha}(x) \ge 0$ for $\alpha \in [-\frac{1}{2}, \frac{1}{2}]$, $x \in [-1, 1]$, $n \in \mathbb{N}$

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$$= \frac{4}{\pi x^2} \begin{cases} (U_{n+1}(x) - xU_n(x))^2 & \text{if } n \text{ is even} \\ (1 - x^2)U_n(x)^2 & \text{if } n \text{ is odd} \end{cases}$$

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Idea: Write

$$S_n^{\alpha}(x) = g_n^{\alpha}(x) - f_n^{\alpha}(x)$$

where $f_n^{\alpha}(x)$ is a sum expression that vanishes for $\alpha = \pm 1/2$, and $g_n^{\alpha}(x)$ is a closed form expression.

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This can be done in many ways.

A good choice turned out to be

$$\begin{aligned} f_n^{\alpha}(x) &= \sum_{k=0}^{2n} \frac{4^{-\alpha}(1-4\alpha^2)}{(2\alpha+2k-1)(2\alpha+2k+3)} \frac{\binom{2\alpha+k}{\alpha}}{\binom{\alpha+k}{\alpha}} P_k^{(\alpha,\alpha)}(0) P_k^{(\alpha,\alpha)}(x) \\ g_n^{\alpha}(x) &= 2^{-2\alpha-1}(2n+1) \frac{\binom{2\alpha+2n+1}{\alpha}}{\binom{\alpha+2n}{\alpha}} P_{2n}^{(\alpha,\alpha)}(0) \\ &\times \left(x P_{2n+1}^{(\alpha,\alpha)}(x) - \frac{2(\alpha+2n+1)}{2\alpha+4n+3} P_{2n}^{(\alpha,\alpha)}(x) \right) \end{aligned}$$

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Indeed, for this choice we have

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 and $f_n^{\pm 1/2}(x)=0.$

This can be verified (but not discovered!) by symbolic summation.

Now $S_n^{\alpha}(x) \ge 0$ is equivalent to $g_n^{\alpha}(x) \ge f_n^{\alpha}(x)$.
































































Now $S_n^{\alpha}(x) \ge 0$ is equivalent to $g_n^{\alpha}(x) \ge f_n^{\alpha}(x)$. Consider $g_n^0(x)$ and $f_n^0(x)$ for $n = 0, \dots, 15$.

The closed form part $g_n^{\alpha}(x)$ contains the main oscillations.



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The closed form part $g_n^{\alpha}(x)$ contains the main oscillations.

So maybe the sum part $f_n^{\alpha}(x)$ is now easier to handle.



Next goal: Find $e_n^{\alpha}(x)$ in closed form such that

 $f_n^{\alpha}(x) \le e_n^{\alpha}(x).$

Consider

$$f_n^{\alpha}(x,y) = \sum_{k=0}^{2n} \frac{4^{-\alpha}(1-4\alpha^2)}{(2\alpha+2k-1)(2\alpha+2k+3)} \frac{\binom{2\alpha+k}{\alpha}}{\binom{\alpha+k}{\alpha}} P_k^{(\alpha,\alpha)}(x) P_k^{(\alpha,\alpha)}(y)$$

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Then $f_n^{\alpha}(x) = f_n^{\alpha}(x, 0).$

It can be shown without too much effort that

$$f_n^{\alpha}(x,y) \le \frac{1}{2} \left(f_n^{\alpha}(x,x) + f_n^{\alpha}(y,y) \right) \quad (\alpha \in \left[-\frac{1}{2}, \frac{1}{2} \right])$$

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There are closed forms for the sums $f_n^\alpha(x,x)$ and $f_n^\alpha(y,y).$ So we may set

$$e_n^{\alpha}(x) := \frac{1}{2} \left(f_n^{\alpha}(x, x) + f_n^{\alpha}(0, 0) \right).$$

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Looks promising...

We have

$$g_{n}^{\alpha}(x) = 2^{-2\alpha-1}(2n+1)\frac{\binom{2\alpha+2n+1}{\alpha}}{\binom{\alpha+2n}{\alpha}}P_{2n}^{(\alpha,\alpha)}(0) \\ \times \left(xP_{2n+1}^{(\alpha,\alpha)}(x) - \frac{2(\alpha+2n+1)}{2\alpha+4n+3}P_{2n}^{(\alpha,\alpha)}(x)\right) \\ e_{n}^{\alpha}(x) = 2^{-2\alpha-1}(2n+1)\frac{\binom{2\alpha+2n+1}{\alpha}}{\binom{\alpha+2n}{\alpha}} \\ \times \left(xP_{2n}^{(\alpha,\alpha)}(x)P_{2n+1}^{(\alpha,\alpha)}(x) - \frac{\alpha+2n+1}{2\alpha+4n+3}P_{2n}^{(\alpha,\alpha)}(x)^{2} - \frac{\alpha+2n+1}{2\alpha+4n+3}P_{2n}^{(\alpha,\alpha)}(0)^{2} - \frac{(2n+1)(2\alpha+2n+1)}{(\alpha+2n+1)(2\alpha+4n+1)}P_{2n+1}^{(\alpha,\alpha)}(x)^{2}\right)$$

We have

$$\begin{split} g_n^{\alpha}(x) &= 2^{-2\alpha - 1} (2n+1) \frac{\binom{2\alpha + 2n + 1}{\alpha}}{\binom{\alpha + 2n}{\alpha}} P_{2n}^{(\alpha, \alpha)}(0) \\ &\times \left(x P_{2n+1}^{(\alpha, \alpha)}(x) - \frac{2(\alpha + 2n + 1)}{2\alpha + 4n + 3} P_{2n}^{(\alpha, \alpha)}(x) \right) \\ e_n^{\alpha}(x) &= 2^{-2\alpha - 1} (2n+1) \frac{\binom{2\alpha + 2n + 1}{\alpha}}{\binom{\alpha + 2n}{\alpha}} \\ &\times \left(x P_{2n}^{(\alpha, \alpha)}(x) P_{2n+1}^{(\alpha, \alpha)}(x) - \frac{\alpha + 2n + 1}{2\alpha + 4n + 3} P_{2n}^{(\alpha, \alpha)}(x)^2 \right. \\ &\left. - \frac{\alpha + 2n + 1}{2\alpha + 4n + 3} P_{2n}^{(\alpha, \alpha)}(0)^2 - \frac{(2n + 1)(2\alpha + 2n + 1)}{(\alpha + 2n + 1)(2\alpha + 4n + 1)} P_{2n+1}^{(\alpha, \alpha)}(x)^2 \right) \end{split}$$

It remains to show $g_n^{\alpha}(x) \ge e_n^{\alpha}(x)$.

After some simplifications, it remains to show

$$\begin{split} (\alpha+2n+1)^2(2\alpha+4n+1)(P_{2n}^{(\alpha,\alpha)}(0)^2+P_{2n}^{(\alpha,\alpha)}(x)^2) \\ &+(2n+1)(2\alpha+2n+1)(2\alpha+4n+3)P_{2n+1}^{(\alpha,\alpha)}(x)^2 \\ &-(\alpha+2n+1)(2\alpha+4n+1) \\ &\times(2\alpha+4n+3)xP_{2n}^{(\alpha,\alpha)}(x)P_{2n+1}^{(\alpha,\alpha)}(x)\geq 0 \end{split}$$
 for $-1\leq x\leq 1$ and $-\frac{1}{2}\leq \alpha\leq \frac{1}{2}.$

After some simplifications, it remains to show

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This would still be a hard thing to do by hand. (Try it.) But CAD and induction on n is applicable here. A Tarski formula for the induction step is...

 $\forall n, \alpha, x, y, z, w \left((n \ge 0 \land -1 \le x \le 1 \land -1 \le 2\alpha \le 1 \land (2\alpha + 4n + 1)(y^2 + z^2)(\alpha + 2n + 1)^2 - (\alpha + 2\alpha + 1)(\alpha + 1)(\alpha + 2\alpha + 1)(\alpha + 2\alpha$ $(2\alpha + 4n + 1)(2\alpha + 4n + 3)wxz(\alpha + 2n + 1) + (2n + 1)(2\alpha + 2n + 1)(2\alpha + 4n + 3)w^2 \ge 0) \Rightarrow$ $(2n+3)(\alpha+2n+1)^2(\alpha+2n+3)^2(2\alpha+2n+3)(2\alpha+4n+5)y^2(\alpha+2n+2)^2 + (\alpha+2n+3)(2\alpha+4n+5)y^2(\alpha+2n+2)^2 + (\alpha+2n+3)(2\alpha+4n+5)y^2(\alpha+2n+3)(2\alpha+4n+5)y^2(\alpha+2n+3)(2\alpha+4n+5)y^2(\alpha+2n+3)(2\alpha+4n+5)y^2(\alpha+2n+3)(2\alpha+4n+5)y^2(\alpha+2n+3)(2\alpha+4n+5)y^2(\alpha+2n+3)(2\alpha+4n+5)y^2(\alpha+2n+3)(2\alpha+4n+5)y^2(\alpha+2n+3)(2\alpha+4n+5)y^2(\alpha+2n+3)(2\alpha+4n+5)y^2(\alpha+2n+3)(2\alpha+4n+5)y^2(\alpha+2n+3)(2\alpha+4n+5)y^2(\alpha+2n+3)(2\alpha+4n+5)y^2(\alpha+2n+3)(2\alpha+4n+5)y^2(\alpha+2n+3)(2\alpha+4n+5)y^2(\alpha+2n+3)(2\alpha+4n+5)y^2(\alpha+2n+3)(2\alpha+4n+5)y^2(\alpha+2n+3)(2\alpha+4n+5)y^2(\alpha+2n+3)(2\alpha+4n+5)y^2(\alpha+2n+3)(2\alpha+4n+5)y^2(\alpha+2n+3)(2\alpha+4n+5)(2\alpha+4n+5)y^2(\alpha+2n+3)(2\alpha+4n+5)y^2(\alpha+2n+3)(2\alpha+4n+5)y^2(\alpha+2n+3)(2\alpha+4n+5)(2\alpha+4n+5)y^2(\alpha+2n+3)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+4n+5)(2\alpha+5$ $1)^{2}(64n^{5} - 256x^{2}n^{4} + 160\alpha n^{4} + 464n^{4} + 144\alpha^{2}n^{3} - 512\alpha x^{2}n^{3} - 1184x^{2}n^{3} + 928\alpha n^{3} + 1344n^{3} + 126x^{2}n^{3} + 126x^{2$ $\begin{array}{c} 56\alpha^3n^2 + 628\alpha^2n^2 - 384\alpha^2x^2n^2 - 1776\alpha x^2n^2 - 1984x^2n^2 + 2016\alpha n^2 + 1944n^2 + 8\alpha^4 n + 164\alpha^3 n + 912\alpha^2 n - 128\alpha^3x^2 n - 888\alpha^2x^2 n - 1984\alpha x^2 n - 1434x^2 n + 1944\alpha n + 1404n + 12\alpha^4 + 120\alpha^3 + 441\alpha^2 - 16\alpha^4x^2 - 148\alpha^3x^2 - 496\alpha^2x^2 - 717\alpha x^2 - 378x^2 + 702\alpha + 405)z^2(\alpha + 2n + 2)^2 - 36\alpha^2x^2 - 717\alpha^2 - 378x^2 + 702\alpha + 405)z^2(\alpha + 2n + 2)^2 - 36\alpha^2x^2 - 717\alpha^2 - 378x^2 + 702\alpha + 405)z^2(\alpha + 2n + 2)^2 - 36\alpha^2x^2 - 717\alpha^2 - 378x^2 + 702\alpha + 405)z^2(\alpha + 2n + 2)^2 - 36\alpha^2x^2 - 717\alpha^2 - 378x^2 + 702\alpha + 405)z^2(\alpha + 2n + 2)^2 - 36\alpha^2x^2 - 717\alpha^2 - 378x^2 + 702\alpha + 405)z^2(\alpha + 2n + 2)^2 - 36\alpha^2x^2 - 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3072x^{2}n^{6} - 896\alpha n^{6} - 1728n^{6} + 12288\alpha x^{4}n^{5} + 25088x^{4}n^{5} - 1216\alpha^{2}n^{5} - 1216\alpha^{$ $\begin{array}{l} w & (-200n + 4096x \ n - 5012x \ n - 5902x \ n - 5902x \ n - 1126n + 112260x \ n + 122002x \ n + 12500x \ n - 1210x \ n - 92166x^2 \ n^5 - 15360x^5 + 15360x^2 \ n^4 + 62720x \ n^4 + 62464x^4 \ n^4 - 800x^3 \ n^4 - 5872a^2 \ n^4 - 11008a^2 \ x^2 \ n^4 - 49920ax^2 \ n^4 - 53120x^2 \ n^4 - 12160a \ n^4 - 7408n^4 - 256a^4 \ n^3 + 10240a^2 \ x^4 \ n^3 + 124928a \ n^4 \ n^3 + 81216a^4 \ n^3 - 3104a^3 \ n^3 - 11072a^2 \ n^3 - 6556a^3 \ x^2 \ n^3 - 14816a \ n^3 - 6592a^3 \ - 326x^2 \ n^2 - 752a^4 \ n^2 + 3340a^4 \ x^4 \ n^2 + 31360a^3 \ x^4 \ n^2 + 93696a^2 \ x^4 \ n^2 + 12182a \ x^4 \ n^2 + 58320x^4 \ n^2 - 4448a^3 \ n^2 - 4448a^$ $10192\alpha^2n^2 - 2112\alpha^4x^2n^2 - 21696\alpha^3x^2n^2 - 76416\alpha^2x^2n^2 - 111264\alpha x^2n^2 - 57396x^2n^2 - 10192\alpha^2n^2 - 111264\alpha x^2n^2 - 10192\alpha^2n^2 -$ $9888\alpha n^2 - 3424n^2 - 64\alpha^5 n - 736\alpha^4 n + 768\alpha^5 x^4 n + 7840\alpha^4 x^4 n + 31232\alpha^3 x^4 n + 60912\alpha^2 x^4 n$ $58320\alpha x^4 n + 21978 x^4 n - 2784\alpha^3 n - 4576\alpha^2 n - 320\alpha^5 x^2 n - 4608\alpha^4 x^2 n - 23296\alpha^3 x^2 n - 4608\alpha^4 x^2 n - 23296\alpha^3 x^2 n - 33296\alpha^3 x^2 n - 33296\alpha$ $53568\alpha^2x^2n - 57396\alpha x^2n - 23340x^2n - 3424\alpha n - 960n - 32\alpha^5 - 240\alpha^4 + 64\alpha^6 x^4 + 784\alpha^5 x^4 + 3904\alpha^4 x^4 + 10152\alpha^3 x^4 + 14580\alpha^2 x^4 + 10989\alpha x^4 + 3402x^4 - 640\alpha^3 - 800\alpha^2 - 16\alpha^6 x^2 - 352\alpha^5 x^2 - 360\alpha^2 + 360\alpha^2 - 360\alpha^2 + 360\alpha^2$ $2504\alpha^{4}x^{2} - 8240\alpha^{3}x^{2} - 13881\alpha^{2}x^{2} - 11670\alpha x^{2} - 3897x^{2} - 480\alpha - 112)(\alpha + 2n + 2)^{2} - 2(\alpha + 2n + 2)^{$ $1)wx(128n^{6} - 1024x^{2}n^{5} + 384\alpha n^{5} + 1408n^{5} + 448\alpha^{2}n^{4} - 2560\alpha x^{2}n^{4} - 5504x^{2}n^{4} + 3520\alpha n^{4} + 35$ $\begin{array}{c} -1.5592n^4 + 256\alpha^3 n^3 + 3296\alpha^2 n^3 - 2560\alpha^2 x^2 n^3 - 1108\alpha x^2 n^3 - 11488x^2 n^3 + 11184\alpha n^3 + 10888n^3 + \\ 72\alpha^4 n^2 + 1424\alpha^3 n^2 + 7870\alpha^2 n^2 - 1280\alpha^3 x^2 n^2 - 8256\alpha^2 x^2 n^2 - 17232\alpha x^2 n^2 - 11688x^2 n^2 + \end{array}$ $\begin{array}{c} 123\alpha^{n} + 1424\alpha^{n} + 17610\alpha^{n} + 1780\alpha^{n} + 1723\alpha^{4}n + 278\alpha^{3}n + 7692\alpha^{2}n - 17232\alpha^{3}n - 11050x^{n} + 11258\alpha^{n} + 11258\alpha^{n} + 11258\alpha^{n} + 11258\alpha^{n} + 11258\alpha^{n} + 7692\alpha^{2}n - 320\alpha^{4}x^{2}n - 2752\alpha^{3}x^{2}n - 8616\alpha^{2}x^{2}n - 11688\alpha x^{2}n - 5814x^{2}n + 11258\alpha n + 5940n + 16\alpha^{5} + 220\alpha^{4} + 1124\alpha^{3} + 2669\alpha^{2} - 32\alpha^{5}x^{2} - 344\alpha^{4}x^{2} - 1436\alpha^{3}x^{2} - 2922\alpha^{2}x^{2} - 2907\alpha x^{2} - 1134x^{2} + 2970\alpha + 1257)z(\alpha + 2n + 2)^{2} \ge 0 \end{array}$

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Pillwein's Proof

Message:

A special function inequality may require some very non-obvious manipulation before an induction proof via CAD succeeds.





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- ... because these inequalities are *soo* difficult.