A Symbolic Decision Procedure for Termination of Linear Programs

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Linear Programs

Consider the termination of the following loop,

$$\mathtt{P1}: \mathtt{while}\ (Bx > b)\ \{\, x := Ax + c \,\}$$

where A is an $n \times n$ matrix, B is an $m \times n$ matrix, and x, b and c are vectors.

Homogeneous loop:

$$\mathtt{P2}: \mathtt{while}\ (Bx > 0)\ \{\, x := Ax \,\}$$

A decision procedure with numerical computation

To determine whether $BA^n x > 0$, we need a clear description of A^n. So, it's natural to consider the *Jordan form* of A.

Theorem (Tiwari'04)

If the program P2 is non-terminating, then there must be a real eigenvector v of A corresponding to a positive eigenvalue such that $Bv \geq 0$.

A decision procedure with numerical computation

Corollary (Tiwari'04)

Assume that for every real eigenvector v of A corresponding to a positive eigenvalue, every element of Bv is not zero. Then, program `P2` is nonterminating if and only if there is a real eigenvector v of A corresponding to a positive eigenvalue such that $Bv > 0$.

A decision procedure with numerical computation

So, if the Jordan form of A is

$$A^* = Q^{-1}AQ$$

and

$$B^* = BQ,$$

then P2 terminates if and only if

$$\text{while } (B^*x > 0)\ \{x := A^*x\}$$

terminates. And by the Theorem (Tiwari'04), we need only to consider the submatrices corresponding to the positive eigenvalues.

A decision procedure with numerical computation

Floating-point computation is needed in the procedure since it depends on the computation of eigenvalues, eigenvectors and Jordan forms.

However, floating-point computation is a source of run-time errors which may lead to a wrong conclusion.

An example

Let

$$A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & b \\ -1 & b \end{pmatrix},$$

where

$$b = -\frac{1127637245}{651041667} = -\sqrt{3} + \epsilon \doteq -1.732050807$$

with $\epsilon = \sqrt{3} - \dfrac{1127637245}{651041667} > 0.$

An example

The approximate eigenvalues of A are 3.732050808 and 0.267949192 (both carry 10 decimal digits of precision). Hence, the Jordan form of A is

$$A^* = Q^{-1}AQ = \begin{pmatrix} 3.732050808 & 0 \\ 0 & 0.267949192 \end{pmatrix}$$

and

$$B^* = BQ = \begin{pmatrix} 1.0 & 0.0 \\ 0.0 & -1.0 \end{pmatrix}.$$

An example

If we let $x = (1, -1)$, after n times of iteration, the loop condition is

$$B^*(A^*)^n x = (3.732050808^n, 0.267949192^n) > (0, 0)$$

which is always true for all n. Therefore, the loop is not terminating.

However, this conclusion is NOT correct.

An example

The Jordan form of A is indeed (by symbolic computation)

$$J = P^{-1}AP = \begin{pmatrix} 2+\sqrt{3} & 0 \\ 0 & 2-\sqrt{3} \end{pmatrix},$$

and,

$$BP = \begin{pmatrix} 1-\frac{\epsilon}{6}\sqrt{3} & \frac{\epsilon}{6}\sqrt{3} \\ -\frac{\epsilon}{6}\sqrt{3} & -1+\frac{\epsilon}{6}\sqrt{3} \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

where $\epsilon = \sqrt{3} - \dfrac{1127637245}{651041667} > 0.$

An example

Obviously, $m_{12} > 0$, $m_{21} < 0$. However, when we use floating-point computation, these two elements (m_{12} and m_{21}) are evaluated to 0 (in Maple 11 with Digits 10). That is why we obtained wrong result by floating-point computation.

Our main results

Theorem Suppose A and B are both matrices on the rational numbers $\mathbb{Q}$ and the characteristic polynomial $D(\lambda)$ of A is irreducible in $\mathbb{Q}[\lambda]$. The program `P2` is nonterminating if and only if there is a real eigenvector v of A corresponding to a positive eigenvalue such that $Bv > 0$.

Our main results

Corollary If A and B are both matrices on a field of numbers (e.g., the second extension of the field of rational numbers) and the characteristic polynomial of A is irreducible on this field, then the program `P2` is nonterminating if and only if there exists a real eigenvector v of A, corresponding to a positive eigenvalue, such that $Bv > 0$.

A symbolic decision procedure

Step 1. Compute the characteristic polynomial of A and denote it by $D(\lambda)$.

Step 2. Compute the algebraic complement minor of every element in the first (or a fixed) row of $A - \lambda I$ (the characteristic matrix of A), respectively and denote them by A_{1i} $(1 \le i \le n)$.

A symbolic decision procedure

Step 3. For each row of B, compute

$$u_j = \sum_{k=1}^{n} b_{jk} A_{1k} \ (1 \leq j \leq m).$$

Step 4. Construct a semi-algebraic system

$$S : \{D(\lambda) = 0,\ \lambda > 0,\ u_1 u_2 > 0,\ \cdots, u_{m-1} u_m > 0\}.$$

A symbolic decision procedure

By computer algebra tools, one can determine, according to the rational coefficients of S, whether S has real solutions. If yes, P2 is not terminating. Otherwise, it terminates.

A brief introduction of DISCOVERER

A semi-algebraic system (SAS) is a system of

$$\left\{\begin{array}{l} p_1(\mathbf{u},\mathbf{x}) = 0, \ldots, p_r(\mathbf{u},\mathbf{x}) = 0, \\ g_1(\mathbf{u},\mathbf{x}) \ge 0, \ldots, g_k(\mathbf{u},\mathbf{x}) \ge 0, \\ g_{k+1}(\mathbf{u},\mathbf{x}) > 0, \ldots, g_t(\mathbf{u},\mathbf{x}) > 0, \\ h_1(\mathbf{u},\mathbf{x}) \ne 0, \ldots, h_m(\mathbf{u},\mathbf{x}) \ne 0, \end{array}\right.$$

where $\mathbf{u} = (u_1, \ldots, u_d)$, $\mathbf{x} = (x_1, \ldots, x_s)$, $r, s \ge 1, t \ge k \ge 0$, $m \ge 0$ and all p_i's, g_i's and h_i's are polynomials in $\mathbb{Q}$.

Parametric SASs and *constant* SASs.

A brief introduction of DISCOVERER

Real Solution Classification of Parametric SASs

For a parametric SAS T and an argument N, DISCOVERER provides **tofind** and **Tofind** to determine the conditions on $\mathbf{u}$ such that the number of the distinct real solutions of T equals N if N is an integer, otherwise falls in the scope N if N is a range like $b..c$ or $b..+\infty$.

A brief introduction of DISCOVERER

Real Solution Isolation of Constant SASs

For a constant SAS T, if T has only a finite number of real solutions, DISCOVERER can determine the number of distinct real solutions of T, say n, and moreover, can find out n disjoint cubes with rational vertices in each of which there is only one solution. In addition, the width of the cubes can be less than any given positive real. The two functions are realized by calling **realzeros** and **nearsolve**.

Examples (I)

We illustrate how to use our algorithm and tool to decide the termination of P2 by the following examples.

$$A = \begin{bmatrix} 3 & 1 & 4 & 1 & 5 \\ 9 & 2 & 6 & 5 & 3 \\ 5 & 8 & 9 & 7 & 9 \\ 3 & 2 & 3 & 8 & 4 \\ 6 & 2 & 6 & 4 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -8 & 3 & 2 & -7 \\ 1 & -4 & 1 & 4 & -2 \\ 4 & -2 & 8 & -5 & 7 \end{bmatrix}.$$

The characteristic polynomial of A is irreducible.

Examples (I)

Compute the algebraic complement minors first.

$$A_{11}(\lambda) = -48 + 313\lambda + 8\lambda^2 - 22\lambda^3 + \lambda^4,$$
$$A_{12}(\lambda) = 381 + 243\lambda - 117\lambda^2 + 9\lambda^3,$$
$$A_{13}(\lambda) = 74 - 539\lambda + 82\lambda^2 + 5\lambda^3,$$
$$A_{14}(\lambda) = 144 - 60\lambda + 15\lambda^2 + 3\lambda^3,$$
$$A_{15}(\lambda) = -498 + 204\lambda - 54\lambda^2 + 6\lambda^3.$$

Examples (I)

Then, compute

$$u_1 = 3A_{11}(\lambda) - 8A_{12}(\lambda) + 3A_{13}(\lambda) + 2A_{14}(\lambda) - 7A_{15}(\lambda)$$
$$= 804 - 4170\lambda + 1614\lambda^2 - 159\lambda^3 + 3\lambda^4,$$
$$u_2 = A_{11}(\lambda) - 4A_{12}(\lambda) + A_{13}(\lambda) + 4A_{14}(\lambda) - 2A_{15}(\lambda)$$
$$= 74 - 1846\lambda + 726\lambda^2 - 53\lambda^3 + \lambda^4,$$
$$u_3 = 4A_{11}(\lambda) - 2A_{12}(\lambda) + 8A_{13}(\lambda) - 5A_{14}(\lambda) + 7A_{15}(\lambda)$$
$$= -4568 - 1818\lambda + 469\lambda^2 - 39\lambda^3 + 4\lambda^4.$$

Examples (I)

By our theorem, the program is nonterminating if and only if the following semi-algebraic system has real solutions.

$$\{D(\lambda) = 0,\ \lambda > 0,\ u_1u_2 > 0,\ u_2u_3 > 0\},$$

where $D(\lambda)$ is the characteristic polynomial of A.

Examples (I)

Using DISCOVERER, say **nearsolve**, we can conclude that the SAS has no real solutions. Therefore, the program is terminating.

If we delete the third row of B (i.e., delete a constraint), by calling DISCOVERER, we can conclude that the system $\{D(\lambda) = 0, \lambda > 0, u_1 u_2 > 0\}$ has 2 distinct real solutions. That is to say, the revised program is nonterminating.

Examples (II)

The following example shows how to generate parametric conditions for the termination of `P2` according to our main theorem.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ \cdots & \cdots & \cdots \\ b_{m1} & b_{m2} & b_{m3} \end{bmatrix}$$

are two matrices on $\mathbb{Q}$ and the characteristics polynomial of A is assumed to be irreducible in $\mathbb{Q}$.

Examples (II)

By following the steps of our algorithm, we finally have to determine whether the following semi-algebraic system has real solutions.

$$\{D(\lambda) = 0,\ \lambda > 0,\ u_1 u_2 > 0, \cdots, u_{m-1} u_m > 0\},$$

where $D(\lambda)$ is the characteristic polynomial of A and u_i are polynomials in λ.

Examples (II)

This is a parametric semi-algebraic system, and we may use the functions `tofind` and `Tofind` of DISCOVERER to generate the required conditions. However, the system includes only univariate polynomials, and the generalized *Complete Discrimination System* (CDS) of polynomials (Yang-Hou-Zhang'96) is very suitable for it.

Examples (II)

Through the CDS tool in DISCOVERER and other tools, when the characteristic polynomial of A is irreducible and A has a unique real eigenvalue, we can conclude that the program is nonterminating iff

$$\begin{aligned} \det(A) > 0 \wedge (\ (\beta_1 > 0 \wedge \cdots \wedge \beta_m > 0) \vee \\ (\beta_1 < 0 \wedge \cdots \wedge \beta_m < 0)\ ), \end{aligned}$$

where for $k = 1, \ldots, m$, β_k is a polynomial of degree 6 with 12 variables and 86 terms.

Examples (III, reducible case)

When the characteristic polynomial of A is reducible, we can still develop a symbolic decision algorithm, which also transforms the linear program termination problem to the problem of determining whether a semi-algebraic system has real solutions symbolically. However the transformation is more complicated than the one in the previous sections. We use the following example to show the main idea of the transformation.

Examples (III, reducible case)

$$A = \begin{bmatrix} 0 & 0 & 0 & -3 \\ 6 & 2 & 0 & 8 \\ -1 & 0 & -3 & 0 \\ -2 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 28 & 13 & 2 & 10 \\ 1 & 4 & 1 & 4 \\ -1 & -1 & 3 & 5 \end{bmatrix},$$

The characteristic polynomial of A is

$$F(\lambda) = \lambda^4 + \lambda^3 - 12\,\lambda^2 - 9\,\lambda + 42 = (\lambda - 2)(\lambda^3 + 3\,\lambda^2 - 6\,\lambda - 21).$$

It has exactly two positive roots $\lambda_1 = 2.557309\ldots$, $\lambda_2 = 2$, which both are of multiplicity 1.

Examples (III, reducible case)

Suppose the corresponding eigenvectors are respectively

$$v_1 = \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \\ v_{14} \end{bmatrix}, \qquad v_2 = \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \\ v_{24} \end{bmatrix} = \begin{bmatrix} 0 \\ 28 \\ 0 \\ 0 \end{bmatrix},$$

and $J = P^{-1}AP$ is the real Jordan form of A.

Examples (III, reducible case)

Then v_1 and v_2 are exactly the two columns of P corresponding to the two positive eigenvalues λ_1 and λ_2. Please note that this claim is not valid for the case that some of the eigenvalues of A is not simple (i.e. not of multiplicity 1).

Examples (III, reducible case)

And for this example, v_1 can be expressed as

$$v_1 = \begin{bmatrix} A_{11}(\lambda_1) \\ A_{12}(\lambda_1) \\ A_{13}(\lambda_1) \\ A_{14}(\lambda_1) \end{bmatrix} = \begin{bmatrix} -\lambda_1^3 - \lambda_1^2 + 6\,\lambda_1 \\ -6\,\lambda_1^2 - 2\,\lambda_1 + 56 \\ \lambda_1^2 - 2\,\lambda_1 \\ 2\,\lambda_1^2 + 3\,\lambda_1 - 14 \end{bmatrix}.$$

Examples (III, reducible case)

The loop P2 is terminating if and only if the following loop P4 is terminating.

$$\texttt{P4}: \quad \texttt{while} \quad (\tilde{B}y > 0) \quad \{\, y := \tilde{J}y \,\},$$

where

$$\tilde{B} = B \times [v_1\, v_2] = \begin{bmatrix} -28\,\lambda_1^3 - 84\,\lambda_1^2 + 168\,\lambda_1 + 588 & 364 \\ -\lambda_1^3 - 16\,\lambda_1^2 + 8\,\lambda_1 + 168 & 112 \\ \lambda_1^3 + 20\,\lambda_1^2 + 5\,\lambda_1 - 126 & -28 \end{bmatrix},$$

$$\tilde{J} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & 2 \end{bmatrix}, \qquad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Examples (III, reducible case)

Obviously, whether the loop `P4` terminates can be decided by whether there exists a real vector y such that for all positive integer n the following inequality holds.

$$\tilde{B} \begin{bmatrix} \lambda_1^n y_1 \\ 2^n y_2 \end{bmatrix} > 0.$$

Examples (III, reducible case)

Firstly, we consider the first column of $\tilde{B}$. Set

$$\begin{aligned} u_1 &= -28(\lambda_1^3 + 3\,\lambda_1^2 - 6\,\lambda_1 - 21), \\ u_2 &= -\lambda_1^3 - 16\,\lambda_1^2 + 8\,\lambda_1 + 168, \\ u_3 &= \lambda_1^3 + 20\,\lambda_1^2 + 5\,\lambda_1 - 126. \end{aligned}$$

It is easy to see that $u_1 = 0$. We then determine whether u_2 and u_3 have the same sign by checking whether the following semi-algebraic system has real solution,

$$\{F(\lambda_1) = 0, \quad \lambda_1 > 2, \quad u_2\,u_3 > 0\}.$$

Examples (III, reducible case)

Using DISCOVERER, e.g., calling `realzeros`, we obtain an output immediately: the system has one real solution which is in $[5/2, 21/8]$. Therefore, if n is large enough, the second and the third inequalities of

$$\tilde{B}\begin{bmatrix} \lambda_1^n y_1 \\ 2^n y_2 \end{bmatrix} > 0.$$

must hold for any (y_1, y_2) with $y_1 > 0$.

Examples (III, reducible case)

Secondly, we consider the second column of $\tilde{B}$. Because we only need to consider the first element of the column and it is positive, the first inequality of

$$\tilde{B}\begin{bmatrix} \lambda_1^n y_1 \\ 2^n y_2 \end{bmatrix} > 0.$$

holds for any (y_1, y_2) with $y_2 > 0$. Therefore, we conclude that the loop does not terminate. For example, the loop does not terminate on $(1, 1)$.

Discussion

1. Any decision procedure that involves floating-point calculation is not feasible in terms of implementation. This paper is to develop a fully symbolic decision procedure for termination of linear program, so that we can avoid floating-point computations in termination analysis. Another interesting issue is how to guarantee a (proved) terminating linear program will indeed terminate, when it runs under a compiler with certain precision of floating point computation.

Discussion

2. One may ask whether the assumption in our main theorem ("the characteristic polynomial of A is irreducible") can be deleted or weakened as "the characteristic polynomial of A is square-free". The answer is negative.

Discussion

3. When the characteristic polynomial of *A* is reducible, we have used an example to explain why we still can avoid computing Jordan forms. The idea is also to transform the termination decision problem to the problem of determining whether a semi-algebraic system has real solutions symbolically as we have used for the irreducible case. However, it is unfortunately much more complicated and the technical details will be presented in another paper.

Thank you !