Math. Num. Sin.
16:1(1994), 108–113.

Chinese J. Num. Math. & Appl.
16:2(1994), 91–96.

# THE GENERALIZED KORN INEQUALITY ON NONCONFORMING FINITE ELEMENT SPACES*

Wang Ming
(*Department of Mathematics, Peking University, Beijing*)

**Abstract**

In this paper, Korn inequality is generalized to the non-conforming finite element spaces, provided the spaces satisfy some conditions.

Poincarè inequality, Poincarè-Friedrichs inequality and Friedrichs inequality play an important role in the existence of solutions of boundary value problems. Korn inequality does the same for the boundary value problems of linear elasticity. The first three inequalities have been generalized to nonconforming and quasi-conforming finite element spaces and applied to the convergence discussion. For some convergent nonconforming element, the generalized Korn inequality is not true[4]. This paper will show that the generalized Korn inequality holds for nonconforming element spaces which satisfy certain conditions.

Let $\Omega$ be a polyhedron domain in $R^n$ with boundary $\partial\Omega$. Denote Sobolev space, Sobolev norm and Sobolev semi-norm by $H^m(\Omega), H_0^m(\Omega)$, $\|\cdot\|_{m,\Omega}$ and $|\cdot|_{m,\Omega}$ respectively.

For element $\mathbf{u}=(u_1,u_2,\cdots,u_n)$ in $(H^m(\Omega))^n$, define

$$\|\mathbf{u}\|_{m,\Omega}=\Big(\sum_{i=1}^n\|u_i\|_{m,\Omega}^2\Big)^{1/2},\quad |\mathbf{u}|_{m,\Omega}=\Big(\sum_{i=1}^n|u_i|_{m,\Omega}^2\Big)^{1/2}.$$

Set

$$\varepsilon_{ij}(\mathbf{u})=\frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\Big),\quad 1\le i,j\le n,$$

when $m\ge 1$. The following Korn inequalities,

$$\|\mathbf{u}\|_{1,\Omega}^2\le C\Big(\sum_{i,j=1}^n\|\varepsilon_{ij}(\mathbf{u})\|_{0,\Omega}^2+\|\mathbf{u}\|_{0,\Omega}^2\Big),\quad \forall\mathbf{u}\in(H^1(\Omega))^n, \tag{1}$$

* Received March 3, 1993.

$$\|\mathbf{u}\|_{1,\Omega}^2 \le C\Big(\sum_{i,j=1}^{n} \|\varepsilon_{ij}(\mathbf{u})\|_{0,\Omega}^2 + \sum_{i=1}^{n}\int_{\partial\Omega} |u_i|^2 ds\Big), \quad \forall \mathbf{u} \in (H^1(\Omega))^n, \tag{2}$$

$$\|\mathbf{u}\|_{1,\Omega}^2 \le C \sum_{i,j=1}^{n} \|\varepsilon_{ij}(\mathbf{u})\|_{0,\Omega}^2, \quad \forall \mathbf{u} \in (H_0^1(\Omega))^n \tag{3}$$

hold. Their proof can be found in [2] or [3]. They will be generalized to nonconforming finite element spaces in this paper.

For parameter $h$, let $\mathsf{T}_h$ be a finite subdivision of $\Omega$. The element in $\mathsf{T}_h$ is an $n$-simplex or $n$-parallelotpe. For $\forall T \in \mathsf{T}_h$, denote the diameter of $T$ by $h_T$ and the inner diameter by $\rho_T$. Assume that there exists a constant $\eta$ independent of $h$, such that $\eta h \le \rho_T < h_T \le h$ for $T \in \mathsf{T}_h$.

Let $V_h \subset L^2(\Omega)$ be a finite element space corresponding to $\mathsf{T}_h$, which satisfies the following conditions,

**H1.** There exists an integer $r \ge 1$, such that, for $\forall v \in V_h$, $v|_T \in P_r(T), \forall T \in \mathsf{T}_h$. Here $P_r(T)$ is the space consisting of all polynomials with degree not greater than $r$.

**H2.** For $\forall v \in V_h, T \in \mathsf{T}_h$, let $F$ be a arbitrary $(n-1)$ dimensional face of $T$, then $v$ is continuous on the set consisting of at least $n$ points on $F$ which are not on a same $(n-2)$ dimensional hyperplane and they are affine invariant.

For $V_h$ , let $V_h^0$ be a subspace of $V_h$, the element $v \in V_h^0$ of which satisfies that $v$ is zero on the set consisting of at least $n$ points on $F$ which are not on a same $(n-2)$ dimensional hyperplane, when $F$ is a $(n-1)$ dimensional face of $T$ in $\mathsf{T}_h$ and $F \subset \partial\Omega$.

For $v \in H^m(\Omega) + V_h$, define

$$\|v\|_{m,h} = \Big(\sum_{T\in\mathsf{T}_h} \|v\|_{m,T}^2\Big)^{1/2}, \quad |v|_{m,h} = \Big(\sum_{T\in\mathsf{T}_h} |v|_{m,T}^2\Big)^{1/2}.$$

For $\mathbf{u} \in (H^m(\Omega) + V_h)^n$, set

$$\|\mathbf{u}\|_{m,h} = \Big(\sum_{i=1}^{n} \|u_i\|_{m,h}^2\Big)^{1/2}, \quad |\mathbf{u}|_{m,h} = \Big(\sum_{i=1}^{n} |u_i|_{m,h}^2\Big)^{1/2}.$$

**Theorem 1.** *Let* H1 *and* H2 *be true. Then there exists a constant $C$ independent of $h$, such that, the following generalized Korn inequalities,*

$$\|\mathbf{v}\|_{1,h}^2 \le C\Big(\sum_{i,j=1}^{n} \sum_{T\in\mathsf{T}_h} \|\varepsilon_{ij}(\mathbf{v})\|_{0,T}^2 + \|\mathbf{v}\|_{0,\Omega}^2\Big), \quad \forall \mathbf{v} \in (V_h)^n, \tag{4}$$

$$\|\mathbf{v}\|_{1,h}^2 \le C\Big(\sum_{i,j=1}^{n} \sum_{T\in\mathsf{T}_h} \|\varepsilon_{ij}(\mathbf{v})\|_{0,T}^2 + \sum_{i=1}^{n}\int_{\partial\Omega} |v_i|^2 ds\Big), \quad \forall \mathbf{v} \in (V_h)^n, \tag{5}$$

$$\|\mathbf{v}\|_{1,h}^2 \le C \sum_{i,j=1}^{n} \sum_{T\in \mathsf{T}_h} \|\varepsilon_{ij}(\mathbf{v})\|_{0,T}^2, \quad \forall \mathbf{v} \in (V_h^0)^n \tag{6}$$

*hold uniformly for* $\forall h$.

To prove the theorem, we need some preliminary results. For $\forall x \in R^n$, let $S_h(x) = \{\, T \mid T \in \mathsf{T}_h$, and $x \in T\}$, and $N_h(x)$ be the number of elements in $S_h(x)$. obviously, $N_h(x)$ is bounded. For $T \in \mathsf{T}_h$, let $\underline{T}$ be the set of inner points of $T$, $E_T$ be the set of vertices of $T$. For $\forall v \in V_h$, $v^T$ is the continuous extension of $v|_{\underline{T}}$ From $\underline{T}$ to $T$. For $h$, define $E_h^1 = \{\, A \mid A \in \Omega$ and $A \in E_T, T \in \mathsf{T}_h\}$, $E_h^2 = \{\, A \mid A \in \partial\Omega$ and $A \in E_T, T \in \mathsf{T}_h\}$.

**Lemma 1.** *Let* H1 *and* H2 *be true. Then there exists a constant* $C$ *independent of* $x, h$, *such that*

$$\sup_{T',T''\in S_h(x)} \left| v_k^{T'}(x) - v_k^{T''}(x)\right| \le Ch^{1-n/2} \sum_{T\in S_h(x)} \sum_{i,j=1}^{n} |\varepsilon_{ij}(\mathbf{v})|_{0,T}, \quad 1 \le k \le n \tag{7}$$

*are true for* $\forall \mathbf{v} \in (V_h)^n, \forall x \in \overline{\Omega}$ *and* $h$ *uniformly, and*

$$\sup_{T\in S_h(x)} |v_k^T(x)| \le Ch^{1-n/2} \sum_{T\in S_h(x)} \sum_{i,j=1}^{n} |\varepsilon_{ij}(\mathbf{v})|_{0,T}, \quad 1 \le k \le n \tag{8}$$

*are true for* $\forall x \in \partial\Omega, \forall \mathbf{v} \in (V_h^0)^n$ *and* $h$ *uniformly.*

*Proof.* (i) Let $\mathbf{v} \in (V_h)^n, x \in \overline{\Omega}$. If $N_h(x) = 1$, then $S_h(x)$ contains only one element, and (7) is true obviously. When $N_h(x) > 1$, $x$ is on the boundary of the elements in $S_h(x)$. Thus, for $T', T'' \in S_h(x)$, there exist $T_1, \cdots, T_l \in S_h(x)$ such that $T_1 = T', T_l = T''$, and $T_t \cap T_{t+1}$ is an $(n-1)$ dimensional face $F_t$ and $x \in F_t, 1 \le t \le l-1$. Since $\mathbf{v}$ has $n$ continuous points on $F_t$ which are not on a same $(n-2)$ dimensional hyperplane, one can get

$$|v_k^{T_t}(x) - v_k^{T_{t+1}}(x)| \le h^{2-n/2}(|v_k|_{2,T_t} + |v_k|_{2,T_{t+1}}),$$

from the interpolation theory. Therefore,

$$|v_k^{T'}(x) - v_k^{T''}(x)| \le \sum_{t=1}^{l-1} |v_k^{T_t}(x) - v_k^{T_{t+1}}(x)| \le Ch^{2-n/2} \sum_{t=1}^{l-1} (|v_k|_{2,T_t} + |v_k|_{2,T_{t+1}})$$
$$\le Ch^{2-n/2} \sum_{T\in S_h(x)} |v_k|_{2,T}. \tag{9}$$

On the other hand, for $1 \le s, t \le n$,

$$\frac{\partial^2 v_k}{\partial x_s \partial x_t} = \frac{\partial}{\partial x_s}\varepsilon_{kt}(\mathbf{v}) + \frac{\partial}{\partial x_t}\varepsilon_{ks}(\mathbf{v}) - \frac{\partial}{\partial x_k}\varepsilon_{st}(\mathbf{v}). \tag{10}$$

By the inverse inequality of polynomial space, one has

$$|v_k|_{2,T} \le Ch^{-1} \sum_{i,j=1}^{n} |\varepsilon_{ij}(\mathbf{v})|_{0,T}. \tag{11}$$

Combining (9) and (11), one get (7). (7) is proved.

(ii) Now let $x \in \partial\Omega, \mathbf{v} \in (V_h^0)^n, T \in S_h(x)$. If a $(n-1)$ dimensional face $F$ of $T$ is on $\partial\Omega$ and $x \in F$, then

$$|v_k^T(x)| \le Ch^{2-n/2}|v_k|_{2,T} \le Ch^{1-n/2}\sum_{i,j=1}^{n}|\varepsilon_{ij}(\mathbf{v})|_{0,T}. \tag{12}$$

since $v_k^T$ has $n$ zero points on $F$ which are not on a same $(n-2)$ dimensional hyperplane.

If there is no $(n-1)$ dimensional face $F$ of $T$ with $F \subset \partial\Omega$ and $x \in F$, then there exist $T_1, \cdots, T_l \in S_h(x)$, such that $T_t \cap T_{t+1}$ is an $(n-1)$ dimensional face $F_t, x \in F_t, 1 \le t \le l-1$, and an $(n-1)$ dimensional face $F_l \subset T_l \cap \partial\Omega, x \in F_l$. Therefore,

$$|v_k^T(x)| \le \sum_{t=1}^{l-1}|v_k^{T_t}(x) - v_k^{T_{t+1}}(x)| + |v_k^{T_l}(x)|$$

(8) follows from (7) and (12).

Corresponding to $\mathsf{T}_h$, let $W_h = \{w \in H^1(\Omega) | \forall T \in \mathsf{T}_h, w|_T \in P_1(T)$ when $T$ is an $n$-simplex, $w|_T$ is an $n$ linear polynomial when $T$ is an $n$-parallelotpe $\}$. The function in $W_h$ is uniquely determined by their values at all points in $E_h^1 \cup E_h^2$. Let $W_h^0 = \{w \in W_h | w(A) = 0, \forall A \in E_h^2\}$, then $W_h^0 \subset H_0^1(\Omega)$.

For $\forall v \in V_h$, define $\Pi_h v \in W_h$ as follows, for $\forall A \in E_h^1 + E_h^2$,

$$\Pi_h v(A) = \frac{1}{N_h(A)}\sum_{T \in S_h(x)} v^T(A),$$

and define $\Pi_h^0 v \in W_h^0$ by

$$\Pi_h^0 v(A) = \begin{cases} \dfrac{1}{N_h(A)}\displaystyle\sum_{T \in S_h(x)} v^T(A), & A \in E_h^1, \\ 0, & A \in E_h^2. \end{cases}$$

For $\forall \mathbf{v} \in (V_h)^n$, let $\Pi_h \mathbf{v} = (\Pi_h v_1, \cdots, \Pi_h v_n)$, $\Pi_h^0 \mathbf{v} = (\Pi_h^0 v_1, \cdots, \Pi_h^0 v_n)$.

**Lemma 2.** *Let* H1 *and* H2 *be true. Then there exists a constant* $C$ *independent of* $h$, *such that,*

$$\sum_{i=0,1} h^{2i}|\mathbf{v} - \Pi_h \mathbf{v}|_{i,h}^2 + h\sum_{i=1}^{n}\int_{\partial\Omega}|v_i - \Pi_h v_i|^2\,ds \le Ch^2\sum_{T \in \mathsf{T}_h}\sum_{i,j=1}^{n}|\varepsilon_{ij}(\mathbf{v})|_{0,T}^2, \quad \forall \mathbf{v} \in (V_h)^n, \tag{13}$$

$$\sum_{i=0,1} h^{2i}|\mathbf{v} - \Pi_h^0 \mathbf{v}|_{i,h}^2 \le Ch^2\sum_{T \in \mathsf{T}_h}\sum_{i,j=1}^{n}|\varepsilon_{ij}(\mathbf{v})|_{0,T}^2, \quad \forall \mathbf{v} \in (V_h^0)^n \tag{14}$$

*hold for* $\forall h$ *uniformly.*

*Proof.* For $\forall T \in \mathrm{T}_h$, let $P_T$ be the interpolation operator defined as follows. $P_T$ is the linear interpolation operator using the the function values on the vertices when $T$ is an $n$-simplex, and $P_T$ is the $n$ linear one when $T$ is an $n$-parallelotpe. By the interpolation theory (see [1] or [4] ), for $\forall \mathbf{v} \in (V_h)^n$,

$$\sum_{k=1}^{n}\sum_{i=0,1} h^{2i} \sum_{T\in \mathrm{T}_h} |v_k - P_T v_k|_{i,T}^2 \le Ch^4 \sum_{k=1}^{n}\sum_{T\in \mathrm{T}_h} |v_k|_{2,T}^2,$$

$$\sum_{k=1}^{n}\sum_{T\in \mathrm{T}_h}\int_{\partial T} |v_k - P_T v_k|^2 ds \le Ch^3 \sum_{k=1}^{n}\sum_{T\in \mathrm{T}_h} |v_k|_{2,T}^2.$$

From (11), one get

$$\sum_{k=1}^{n}\sum_{i=0,1} h^{2i} \sum_{T\in \mathrm{T}_h} |v_k - P_T v_k|_{i,T}^2 + h\sum_{k=1}^{n}\sum_{T\in \mathrm{T}_h}\int_{\partial T} |v_k - P_T v_k|^2 ds$$

$$\le Ch^2 \sum_{i,j=1}^{n}\sum_{T\in \mathrm{T}_h} |\varepsilon_{ij}(\mathbf{v})|_{0,T}^2. \tag{15}$$

When $T \in \mathrm{T}_h$ is an $n$-simplex, it is easy to prove that $\forall p \in P_1(T)$,

$$|p|_{0,T}^2 + h|p|_{0,\partial T}^2 \le Ch^n \sum_{A\in E_T} |p(A)|^2. \tag{16}$$

(16) is also true for $n$ linear polynomial $p$ when $T$ is an $n$-parallelotpe. From the definitions of $P_T$ and $\Pi_h$, one has

$$\sum_{k=1}^{n}\sum_{T\in \mathrm{T}_h} |P_T v_k - \Pi_h v_k|_{0,T}^2 + h\sum_{k=1}^{n}\sum_{T\in \mathrm{T}_h} |P_T v_k - \Pi_h v_k|_{0,\partial T}^2$$

$$\le Ch^n \sum_{T\in \mathrm{T}_h}\sum_{A\in E_T} |P_T v_k(A) - \Pi_h v_k(A)|^2$$

$$\le Ch^n \sum_{T\in \mathrm{T}_h}\sum_{A\in E_T} |v_k^T(A) - \Pi_h v_k(A)|^2. \tag{17}$$

When $T \in \mathrm{T}_h$, $A \in E_T$, from (7), one derives that

$$|v_k^T(A) - \Pi_h v_k(A)| = \left| v_k^T(A) - \frac{1}{N_h(A)} \sum_{T'\in S_h(A)} v_k^{T'}(A) \right|$$

$$\le \frac{1}{N_h(A)} \sum_{T'\in S_h(A)} |v_k^T(A) - v_k^{T'}(A)|$$

$$\le Ch^{1-n/2} \sum_{T'\in S_h(A)}\sum_{i,j=1}^{n} |\varepsilon_{ij}(\mathbf{v})|_{0,T'}. \tag{18}$$

By (17) and (18) as well as the inverse inequality, one get

$$\sum_{k=1}^{n}\sum_{i=0,1}h^{2i}\sum_{T\in\mathrm{T}_h}|P_Tv_k-\Pi_hv_k|_{i,T}^2+h\sum_{k=1}^{n}\sum_{T\in\mathrm{T}_h}\int_{\partial T}|P_Tv_k-\Pi_hv_k|^2ds$$

$$\le Ch^2\sum_{i,j=1}^{n}\sum_{T\in\mathrm{T}_h}|\varepsilon_{ij}(\mathbf{v})|_{0,T}^2.$$

Inequality (13) follows from (15) and the above inequality. Inequality (14) can be obtained by similar way.

Now we prove the theorem. Let $\mathbf{v}\in(V_h)^n$, then $\Pi_h\mathbf{v}\in(H^1(\Omega))^n$. From (1) and (13), one has

$$\begin{aligned}\|\mathbf{v}\|_{1,h}^2&\le 2(\|\mathbf{v}-\Pi_h\mathbf{v}\|_{1,h}^2+\|\Pi_h\mathbf{v}\|_{1,\Omega}^2)\\&\le C\Big(\sum_{T\in\mathrm{T}_h}\sum_{i,j=1}^{n}|\varepsilon_{ij}(\mathbf{v})|_{0,T}^2+\sum_{i,j=1}^{n}|\varepsilon_{ij}(\Pi_h\mathbf{v})|_{0,\Omega}^2+\|\Pi_h\mathbf{v}\|_{0,\Omega}^2\Big)\\&\le C\Big(\sum_{T\in\mathrm{T}_h}\sum_{i,j=1}^{n}|\varepsilon_{ij}(\mathbf{v})|_{0,T}^2+\sum_{i,j=1}^{n}\sum_{T\in\mathrm{T}_h}|\varepsilon_{ij}(\Pi_h\mathbf{v})-\varepsilon_{ij}(\mathbf{v})|_{0,T}^2\\&\quad+\|\mathbf{v}-\Pi_h\mathbf{v}\|_{0,\Omega}^2+\|\mathbf{v}\|_{0,\Omega}^2\Big)\le C\Big(\sum_{T\in\mathrm{T}_h}\sum_{i,j=1}^{n}|\varepsilon_{ij}(\mathbf{v})|_{0,T}^2+\|\mathbf{v}\|_{0,\Omega}^2\Big)\end{aligned}$$

that is, inequality (4) is true. Similarly, inequality (5) is obtained by (2) and (13), and inequality (6) by (3) and (14).

For the finite element methods of the linear elasticity problem, one can get, by the generalized Korn inequality, that a non-conforming elements, convergent in the energy norm, is also convergent in norm $\|\cdot\|_{1,h}$ with the same error bounds, when $\Omega$ is not convex. For example, Wilson element is convergent in norm $\|\cdot\|_{1,h}$ and the error is $O(h)$.

## References


[1] P.C. Ciarlet, The Finite Element Method for Elliptic Problems, North- Holland, Amesterdam, New York, 1978.

[2] G. Fichera, Existence theorems in elasticity, Handbuch der Physik, Band VI, 12, 347-389, Springer-Verlag, Heidelberg, 1972.

[3] Zhang Hong-qing and Wang Ming, Generalized Korn-Poincare's inequalities and their applications (I), Science Exploration, Vol.2, No.3, 83–92, 1982. (in Chinese)

[4] Zhang Hong-qing and Wang Ming, The Mathematical Theory of Finite Element Methods, Science Press, Beijing, 1991. (in Chinese)