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A New Class of Zienkiewicz-Type Nonconforming Element in Any Dimensions

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Summary In this paper, a new class of Zienkiewicz-type nonconforming finite element, in n spatial dimensions with $n \geq 2$, is proposed. The new finite element is proved to be convergent for the biharmonic equation.

Key words Nonconforming finite element, forth order elliptic equation, biharmonic

1 Introduction

In this paper, we will propose a new class of Zienkiewicz-type nonconforming simplex finite element for n-dimensional fourth order partial differential equations with $n \geq 2$. It uses the values of function and first order derivatives at vertices as degrees of freedom, that is, it uses the same degrees of freedom with the Zienkiewicz element [2 or 6]. But its shape function space is different from the one of the Zienkiewicz element.

As a nonconforming finite element for fourth order partial differential equations, the Zienkiewicz element is attractive. The first thing is its convergent property. In two dimensional case, the Zienkiewicz element is only convergent under the parallel line condition, and is divergent in general grids. The numerical experiments were given in [7] and the mathematical proof can be found in [10]. Another attractive thing is the degrees of freedom of the Zienkiewicz element. It is convenient for numerical computations to take values of function and derivatives at vertices as degrees of freedom. Although on each single element the number of degrees of freedom of the Zienkiewicz element is not the least, the global number is.

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There are some modified triangular elements, with the same degrees of freedom of the Zienkiewicz element, proposed by different ways, such as, the TQC9 element by the quasi-conforming method [18,5], the generalized conforming element by the generalized conforming method [9], the TRUNC element by the free formula method [1,4] and the Bergan element by the energy orthogonal method [3]. We call these elements are of Zienkiewicz-type, Z-type for short. Their convergence analysis were given in [20,13,11,14] respectively. It is a little surprise that there were no convergent Z-type element proposed directly from the nonconforming element method in two dimensions.

In three dimensional case, two convergent Z-type elements were proposed in [19]. One is constructed by the quasi-conforming method, and another is a non C^0 nonconforming element which is reduced from a cubic tetrahedral nonconforming element proposed also in [19]. For this cubic element, the number of the degrees of freedom is just the dimension of the cubic polynomial space. It does not occur in other dimensional cases when the similar degrees of freedom are used.

The new Z-type element proposed in this paper is a nonconforming C^0 element, and it is constructed in a canonical fashion for two and higher dimensions.

The rest of the paper is organized as follows. Section 2 recalls the nonconforming element method. Section 3 gives a detailed descriptions of a new Z-type nonconforming element. Section 4 shows the convergence of the new element. The last section gives some concluding remarks.

2 Preliminaries

Let Ω be a bounded polyhedroid domain in R^n ($n \geq 2$) with boundary $\partial\Omega$. For a nonnegative integer s, Let $H^s(\Omega)$, $H_0^s(\Omega)$, $\|\cdot\|_{s,\Omega}$ and $|\cdot|_{s,\Omega}$ denote the usual Sobolev spaces, norm and semi-norm respectively. Let $(\cdot,\cdot)$ denote the inner product of $L^2(\Omega)$.

For $f \in L^2(\Omega)$, we consider the following fourth order boundary value problem:

$$\begin{cases} \Delta^2 u = f, & \text{in } \Omega, \\ u|_{\partial\Omega} = \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0 \end{cases} \tag{1}$$

where $\nu = (\nu_1, \nu_2, \cdots, \nu_n)^\top$ is the unit outer normal to $\partial\Omega$ and Δ is the standard Laplacian operator.

Set

$$\nabla = \Big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n}\Big)^\top.$$

Define

$$a(v,w) = \int_\Omega \sum_{i,j=1}^n \frac{\partial^2 v}{\partial x_i \partial x_j} \frac{\partial^2 w}{\partial x_i \partial x_j}, \quad \forall v, w \in H^2(\Omega). \tag{2}$$

The weak form of problem (1) is: find $u \in H_0^2(\Omega)$ such that

$$a(u,v) = (f,v), \qquad \forall v \in H_0^2(\Omega). \tag{3}$$

For a subset $B \subset R^n$ and a nonnegative integer r, let $P_r(B)$ be the space of all polynomials with degree not greater than r.

Let (T, P_T, Φ_T) be a finite element where T is the geometric shape, P_T the shape function space and Φ_T the vector of degrees of freedom, and let Φ_T be P_T-unisolvent (see [6]). Let $\mathcal{T}_h$ be a triangulation of Ω with mesh size h. For each element $T \in \mathcal{T}_h$, let h_T be the diameter of the smallest ball containing T and ρ_T be the diameter of the largest ball contained in T.

Let $\{\mathcal{T}_h\}$ be a family of triangulations with $h \to 0$. Throughout the paper, we assume that $\{\mathcal{T}_h\}$ is quasi-uniform, namely it satisfied that $h_T \leq h \leq \eta\rho_T, \forall T \in \mathcal{T}_h$ for a positive constant η independent of h.

For each $\mathcal{T}_h$, let V_h and V_{h0} be the corresponding finite element spaces associated with (T, P_T, Φ_T) for the discretization of $H^2(\Omega)$ and $H_0^2(\Omega)$ respectively. In the case of nonconforming element, $V_h \not\subset H^2(\Omega)$ and $V_{h0} \not\subset H_0^2(\Omega)$.

For $v, w \in L^2(\Omega)$ that $v|_T, w|_T \in H^2(T), \forall T \in \mathcal{T}_h$, we define

$$a_h(v, w) = \sum_{T \in \mathcal{T}_h} \int_T \sum_{i,j=1}^{n} \frac{\partial^2 v}{\partial x_i \partial x_j} \frac{\partial^2 w}{\partial x_i \partial x_j}. \tag{4}$$

The finite element method for problem (3) corresponding to the element (T, P_T, Φ_T) is: find $u_h \in V_{h0}$ such that

$$a_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_{h0}. \tag{5}$$

For any $v \in L^2(\Omega)$ that $v|_T \in H^m(\Omega)$, $\forall T \in \mathcal{T}_h$, we define the following mesh-dependent norm $\|\cdot\|_{m,h}$ and semi-norm $|\cdot|_{m,h}$:

$$\|v\|_{m,h} = \Big(\sum_{T \in \mathcal{T}_h} \|v\|_{m,T}^2 \Big)^{1/2}, \quad |v|_{m,h} = \Big(\sum_{T \in \mathcal{T}_h} |v|_{m,T}^2 \Big)^{1/2}.$$

For nonconforming elements, the basic mathematical theory has been established (see [6,8,15-17]). We will use them to discuss the convergence of the new element.

3 A New Z-Type Nonconforming Element

In this section, we give a detailed description of our new Z-type nonconforming element in n-dimensions ($n \geq 2$).

Given an n-simplex T with vertices a_i, $1 \leq i \leq n+1$, denote by F_i ($1 \leq i \leq n+1$) the $(n-1)$-dimensional subsimplex of T without a_i as its vertex, and by $\lambda_1, \lambda_2, \cdots, \lambda_{n+1}$ the barycentric coordinates of T. Denote by $|T|$ and $|F_i|$ the measures of T and F_i respectively.

Define

$$P_3'(T) = P_2(T) + \mathrm{span}\,\{\lambda_i^2\lambda_j - \lambda_i\lambda_j^2 \mid 1 \leq i < j \leq n+1\}.$$

Then the shape function space of the n-dimensional Zienkiewicz element is just $P_3'(T)$.

Now let q_0 be the bubble function defined by

$$q_0 = \lambda_1 \lambda_2 \cdots \lambda_{n+1}.$$

For $1 \le i < j \le n+1$, we define

$$\begin{aligned} q_{ij} &= \lambda_i^2 \lambda_j - \lambda_i \lambda_j^2 \\ &\quad + \tfrac{(2n-1)!}{n!} \Big(\tfrac{(n-1)n}{n+1} (\lambda_i - \lambda_j) + \sum_{\substack{1 \le k \le n+1 \\ k \ne i, k \ne j}} \tfrac{(\nabla \lambda_i - \nabla \lambda_j)^\top \nabla \lambda_k}{\|\nabla \lambda_k\|^2} (n\lambda_k - 1) \Big) q_0. \end{aligned} \tag{6}$$

A new Z-type nonconforming element, NZT element for short, is defined by (T, P_T, Φ_T) with

1) T is an n-simplex.
2) $P_T = P_2(T) + \mathrm{span}\{\, q_{ij} \mid 1 \le i < j \le n+1 \}$.
3) The components of Φ_T are:

$$v(a_j),\ 1 \le j \le n+1,\ \ (a_j - a_i)^\top \nabla v(a_i),\ 1 \le i \ne j \le n+1, \quad \forall v \in C^1(T).$$

The degrees of freedom of the NZT element are just the same with the n-dimensional Zienkiewicz element (see Fig. 1).

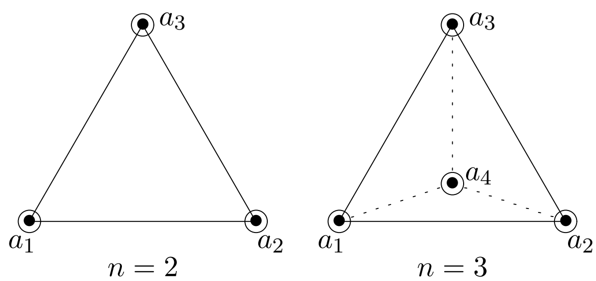

Fig. 1

Let $\nu^{(i)}$ denote the unit out normal of $(n-1)$-subsimplex F_i of T ($1 \le i \le n+1$). By certain computation, we can obtain that

$$\frac{1}{|F_i|} \int_{F_i} \frac{\partial p}{\partial \nu^{(i)}} = \frac{1}{n} \sum_{\substack{1 \le k \le n+1 \\ k \ne i}} \frac{\partial p}{\partial \nu^{(i)}}(a_k), \quad 1 \le i \le n+1,\ \forall p \in P_T. \tag{7}$$

Lemma 1 *For NZT element, Φ_T is P_T-unisolvent.*

Proof Let $p \in P_T$ and

$$p(a_j) = 0,\ 1 \le j \le n+1; \quad (a_j - a_i)^\top \nabla p(a_i) = 0,\ 1 \le i \ne j \le n+1.$$

We only need to show that $p \equiv 0$. Let $q_1, \cdots, q_l$ be a basis of $P_2(T)$. Then p can be wriiten as

$$p = \sum_{i=1}^{l} c_i q_i + \sum_{1 \le i < j \le n+1} c_{ij} q_{ij}$$

where c_i and c_{ij} are constants. Set $\tilde{q}_{ij} = \lambda_i^2\lambda_j - \lambda_i\lambda_j^2$ and

$$\tilde{p} = \sum_{i=1}^{l} c_i q_i + \sum_{1\le i<j\le n+1} c_{ij}\tilde{q}_{ij}.$$

Then

$$p = \tilde{p} + \sum_{1\le i<j\le n+1} c_{ij}(q_{ij} - \tilde{q}_{ij}).$$

It can be verified that

$$(q_{ij}-\tilde{q}_{ij})(a_k) = 0,\ 1\le k\le n+1;\ (a_m-a_k)^\top\nabla(q_{ij}-\tilde{q}_{ij})(a_k) = 0,\ 1\le k\ne m\le n+1.$$

Thus, $\tilde{p}$ satisfies

$$\tilde{p}(a_j) = 0,\ 1\le j\le n+1; \quad (a_j-a_i)^\top\nabla\tilde{p}(a_i) = 0,\ 1\le i\ne j\le n+1.$$

On the other hand, $\tilde{p}\in P_3'(T)$. Thus $\tilde{p}\equiv 0$, that is,

$$c_i = 0,\ 1\le i\le l;\ \ c_{ij} = 0,\ 1\le i<j\le n+1.$$

It follows that $p\equiv 0$.

For $1\le i\ne j\le n+1$ we define

$$\begin{cases} p_{ij} = \dfrac{1}{2}\lambda_i\lambda_j(1+\lambda_i-\lambda_j), \\ \qquad +\frac{(2n-1)!}{2n!}\Big(\frac{(n-1)n}{n+1}(\lambda_i-\lambda_j) + \sum\limits_{\substack{1\le k\le n+1\\ k\ne i,k\ne j}} \frac{(\nabla\lambda_i-\nabla\lambda_j)^\top\nabla\lambda_k}{\|\nabla\lambda_k\|^2}(n\lambda_k-1)\Big)q_0 \\ p_i = \lambda_i^2 + 2\sum\limits_{\substack{1\le j\le n+1\\ j\ne i}} p_{ij}. \end{cases} \tag{8}$$

Let δ_{ij} be the Kronecker delta. It can be verified that for $1\le i\ne j\le n+1$ and $1\le k\ne l\le n+1$,

$$\begin{cases} p_i(a_k) = \delta_{ik},\ (a_l-a_k)^\top\nabla p_i(a_k) = 0, \\ p_{ij}(a_k) = 0,\ \ (a_l-a_k)^\top\nabla p_{ij}(a_k) = \delta_{ik}\delta_{jl}. \end{cases} \tag{9}$$

That is, p_i and p_{ij} are the nodal basis functions respect to the degrees of freedom. The corresponding interpolation operator Π_T can be written by,

$$\Pi_T v = \sum_{1\le i\le n+1} p_i v(a_i) + \sum_{1\le i\ne j\le n+1} p_{ij}(a_j-a_i)^\top\nabla v(a_i), \quad \forall v\in C^1(T). \tag{10}$$

For NZT element, we can define the corresponding finite element spaces V_h and V_{h0} as follows: $V_h = \{v\in L^2(\Omega)\mid v|_T\in P_T, \forall T\in T_h,\ v$ and ∇v are continuous at all vertices of elements in $\mathcal{T}_h\}$. $V_{h0} = \{v\in V_h\mid v$ and ∇v vanish at all vertices belonging to $\partial\Omega\}$.

We claim that $V_h \subset H^1(\Omega)$ and $V_{h0} \subset H_0^1(\Omega)$. Let $v_h \in V_h$, and let F be a common $(n-1)$-dimensional subsimplex of $T, T' \in \mathcal{T}_h$. By definition, the restrictions of $v_h|_T$ and $v_h|_{T'}$ on F are all in $P_3'(F)$, and they and their first order derivatives are equal at all vertices of F respectively. Thus $v_h|_T = v_h|_{T'}$ on F, that is, $v_h \in C^0(\bar{\Omega})$, and this leads to that $v_h \in H^1(\Omega)$. Using similar argument, we can show $v_h \in H_0^1(\Omega)$ when $v_h \in V_{h0}$.

Given any $(n-1)$-dimensional subsimplex F and $v_h \in V_h$, let us define the jump of ∇v_h across F as follows:

$$[\nabla v_h] = \nabla v_h|_T - \nabla v_h|_{T'}$$

if $F = T \cap T'$ for some $T, T' \in \mathcal{T}_h$ and

$$[\nabla v_h] = \nabla v_h|_T$$

if $F = T \cap \partial\Omega$.

The following lemma is a direct consequence of equality (7) and the definitions of V_h and V_{h0}.

Lemma 2 *Let V_h and V_{h0} be the finite element spaces corresponding to NZT element. If F is a common $(n-1)$-dimensional subsimplex of $T, T' \in \mathcal{T}_h$, then*

$$\int_F [\nabla v_h] = 0, \quad \forall v_h \in V_h. \tag{11}$$

If an $(n-1)$-dimensional subsimplex F of $T \in \mathcal{T}_h$ is on $\partial\Omega$ then

$$\int_F [\nabla v_h] = 0, \quad \forall v_h \in V_{h0}. \tag{12}$$

Remark. Let V_h and V_{h0} be the finite element spaces corresponding to NZT element. By Lemma 2 and Green's formula, we can obtain directly that

$$a_h(p, v_h) = 0, \quad \forall p \in P_2(\bar{\Omega}),\ \forall v_h \in V_{h0}.$$

We know from [15] that the NZT element passes the patch test on triangulation $\mathcal{T}_h$.

4 Convergence Analysis

In this section, we discuss the convergence property of NZT element. Let V_h and V_{h0} be the finite element spaces corresponding to NZT element. First, we consider the error estimates for finite element spaces.

Theorem 1 *Let V_h and V_{h0} be the finite element spaces corresponding to NZT element. Then there exists a constant C independent of h such that*

$$\inf_{v_h \in V_{h0}} \sum_{m=0}^{3} h^m |v - v_h|_{m,h} \le C h^3 |v|_{3,\Omega},\ \forall v \in H^3(\Omega) \cap H_0^2(\Omega), \tag{13}$$

$$\inf_{v_h \in V_h} \sum_{m=0}^{3} h^m |v - v_h|_{m,h} \le C h^3 |v|_{3,\Omega},\ \forall v \in H^3(\Omega). \tag{14}$$

Proof For $v \in H^3(\Omega) \cap H_0^2(\Omega)$, let $w_h \in L^2(\Omega)$ such that $\forall T \in \mathcal{T}_h$, $w_h|_T \in P_2(T)$ and

$$\int_T q w_h dx = \int_T q v dx, \quad \forall q \in P_2(T).$$

By the interpolation theory, we have

$$|v - w_h|_{m,h} \le C h^{3-m} |v|_{3,\Omega}, \quad 0 \le m \le 3. \tag{15}$$

Given a set $B \subset R^n$, let $\mathcal{T}_h(B) = \{\, T \in \mathcal{T}_h \mid B \cap T \neq \emptyset \,\}$ and $N_h(B)$ be the number of the elements in $\mathcal{T}_h(B)$.

Now we define $v_h \in V_{h0}$ as follows: for any $T \in \mathcal{T}_h$, if vertex a_i of T is in Ω then

$$v_h(a_i) = \frac{1}{N_h(a_i)} \sum_{T' \in \mathcal{T}_h(a_i)} (w_h|_{T'})(a_i),\ \nabla v_h(a_i) = \frac{1}{N_h(a_i)} \sum_{T' \in \mathcal{T}_h(a_i)} \nabla (w_h|_{T'})(a_i).$$

Obviously, v_h is well-defined. We will show that

$$|v - v_h|_{m,h} \le C h^{3-m} |v|_{3,\Omega}, \quad 0 \le m \le 3. \tag{16}$$

Let $T \in \mathcal{T}_h$, by a standard scaling argument, we have

$$|p|_{m,T}^2 \le C h^{n-2m} \sum_{i=1}^{n+1} \Big(|p(a_i)|^2 + h^2 \|\nabla p(a_i)\|^2 \Big), \quad 0 \le m \le 3,\ \forall p \in P_T. \tag{17}$$

Set $\phi_h = w_h - v_h$. Obviously, $\phi_h|_T \in P_T$.

If vertex a_i of T is in Ω, the definition of v_h leads to that

$$(\phi_h|_T)(a_i) = \frac{1}{N_h(a_i)} \sum_{T' \in \mathcal{T}_h(a_i)} \Big((w_h|_T)(a_i) - (w_h|_{T'})(a_i) \Big).$$

For $T' \in \mathcal{T}_h(a_i)$ there exist $T_1, \cdots, T_J \in \mathcal{T}_h(a_i)$ such that $T_1 = T$, $T_J = T'$ and $\tilde{F}_j = T_j \cap T_{j+1}$ is a common $(n-1)$-dimensional subsimplex of T_j and T_{j+1} and $a_i \in \tilde{F}_j$, $1 \le j < J$. By the inverse inequality, we have

$$\begin{aligned}
\Big| (w_h|_T)(a_i) - (w_h|_{T'})(a_i) \Big|^2 &= \Big| \sum_{j=1}^{J-1} \big((w_h|_{T_j})(a_i) - (w_h|_{T_{j+1}})(a_i) \big) \Big|^2 \\
&\le C \sum_{j=1}^{J-1} \Big| (w_h|_{T_j})(a_i) - (w_h|_{T_{j+1}})(a_i) \Big|^2 \\
&\le C h^{1-n} \sum_{j=1}^{J-1} \Big| w_h|_{T_j} - w_h|_{T_{j+1}} \Big|_{0,\tilde{F}_j}^2 \\
&\le C h^{1-n} \sum_{j=1}^{J-1} \Big(\Big| v - w_h|_{T_j} \Big|_{0,\tilde{F}_j}^2 + \Big| v - w_h|_{T_{j+1}} \Big|_{0,\tilde{F}_j}^2 \Big)
\end{aligned}$$

By the interpolation theory, we obtain

$$\left|(w_h|_T)(a_i) - (w_h|_{T'})(a_i)\right|^2 \le Ch^{6-n} \sum_{j=1}^{J} |v|^2_{3,T_j}$$

Since $N_h(T)$ is bounded, we have

$$|(\phi_h|_T)(a_i)|^2 \le Ch^{6-n} \sum_{T' \in \mathcal{T}_h(T)} |v|^2_{3,T'} \tag{18}$$

If vertex a_i of T is on $\partial\Omega$ then there exists $T' \in \mathcal{T}_h(a_i)$ with an $(n-1)$-dimensional subsimplex F of T' on $\partial\Omega$ and $a_i \in F$. By the definitions of w_h and v_h, we have

$$\begin{aligned}|(\phi_h|_T)(a_i)| &= |(w_h|_T)(a_i) - (w_h|_{T'})(a_i) + (w_h|_{T'})(a_i)| \\ &\le |(w_h|_T)(a_i) - (w_h|_{T'})(a_i)| + |(w_h|_{T'})(a_i)|.\end{aligned}$$

By the inverse inequality and the interpolation theory

$$|(w_h|_{T'})(a_i)|^2 \le Ch^{1-n}|w_h|_{T'}|^2_{0,F} = Ch^{1-n}|v - w_h|_{T'}|^2_{0,F} \le Ch^{6-n}|v|^2_{3,T'}.$$

By a similar analysis for $|(w_h|_T)(a_i) - (w_h|_{T'})(a_i)|$, we conclude that (18) is also true in this case.

Similarly, we can show that

$$\sum_{i=1}^{n+1} \|\nabla(\phi_h|_T)(a_i)\|^2 \le Ch^{4-n} \sum_{T' \in \mathcal{T}_h(T)} |v|^2_{3,T'}. \tag{19}$$

Combining (17), (18) and (19), we have

$$h^{2m}|\phi_h|^2_{m,T} \le Ch^6 \sum_{T' \in \mathcal{T}_h(T)} |v|^2_{3,T'}.$$

Summing the above inequality over all $T \in \mathcal{T}_h$, we obtain that

$$h^{2m}|\phi_h|^2_{m,h} \le Ch^6 \sum_{T \in \mathcal{T}_h} \sum_{T' \in \mathcal{T}_h(T)} |v|^2_{3,T'}.$$

Consequently

$$h^{2m}|\phi_h|^2_{m,h} \le Ch^6|v|^2_{3,\Omega}. \tag{20}$$

Inequality (16) follows from (20) and (15).

Using similar argument, we can show (14).

Lemma 3 *Let V_{h0} be the finite element space corresponding to NZT element. Then there exists a constant C independent of h such that for $v \in H^3(\Omega)$*

$$|a_h(v, v_h) - (\Delta^2 v, v_h)| \le Ch|v|_{3,\Omega}|v_h|_{2,h}, \quad \forall v_h \in V_{h0}. \tag{21}$$

Proof Let $v_h \in V_{h0}$ and $\phi \in H^1(\Omega)$. Given $T \in \mathcal{T}_h$ and an $(n-1)$-dimensional subsimplex F of T, let $P_F^0 : L^2(F) \to P_0(F)$ be the L^2-orthogonal projection.

Let $i, j \in \{1, 2, \cdots, n\}$. By Lemma 2 and Green's formula we have

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} \int_T \Big(\phi \frac{\partial^2 v_h}{\partial x_i \partial x_j} + \frac{\partial \phi}{\partial x_i} \frac{\partial v_h}{\partial x_j}\Big) &= \sum_{T\in\mathcal{T}_h} \int_{\partial T} \phi \frac{\partial v_h}{\partial x_j} \nu_i = \sum_{T\in\mathcal{T}_h} \sum_{F\subset\partial T} \int_F \phi \frac{\partial v_h}{\partial x_j} \nu_i \\ &= \sum_{T\in\mathcal{T}_h} \sum_{F\subset\partial T} \int_F \phi \Big(\frac{\partial v_h}{\partial x_j} - P_F^0 \frac{\partial v_h}{\partial x_j}\Big) \nu_i \\ &= \sum_{T\in\mathcal{T}_h} \sum_{F\subset\partial T} \int_F (\phi - P_F^0 \phi) \Big(\frac{\partial v_h}{\partial x_j} - P_F^0 \frac{\partial v_h}{\partial x_j}\Big) \nu_i\end{aligned}$$

Using the Schwarz inequality and the interpolation theory we obtain that

$$\begin{aligned}&\Big| \sum_{T\in\mathcal{T}_h} \sum_{F\subset\partial T} \int_F (\phi - P_F^0 \phi) \Big(\frac{\partial v_h}{\partial x_j} - P_F^0 \frac{\partial v_h}{\partial x_j}\Big) \nu_i \Big| \\ &\qquad \le \sum_{T\in\mathcal{T}_h} \sum_{F\subset\partial T} \|\phi - P_F^0 \phi\|_{0,F} \Big\| \frac{\partial v_h}{\partial x_j} - P_F^0 \frac{\partial v_h}{\partial x_j} \Big\|_{0,F} \\ &\qquad \le C \sum_{T\in\mathcal{T}_h} h |\phi|_{1,T} |v_h|_{2,T} \le Ch |\phi|_{1,\Omega} |v_h|_{2,h}.\end{aligned}$$

Consequently, for $i, j \in \{1, 2, \cdots, n\}$,

$$\Big| \sum_{T\in\mathcal{T}_h} \int_T \Big(\phi \frac{\partial^2 v_h}{\partial x_i \partial x_j} + \frac{\partial \phi}{\partial x_i} \frac{\partial v_h}{\partial x_j}\Big) \Big| \le Ch |\phi|_{1,\Omega} |v_h|_{2,h}, \quad \forall \phi \in H^1(\Omega),\ \forall v_h \in V_{h0}. \tag{22}$$

Using (22) and the following equality,

$$\begin{aligned}a_h(v, v_h) - (\Delta^2 v, v_h) &= \sum_{i=1}^n \sum_{T\in\mathcal{T}_h} \int_T \Big(\Delta v \frac{\partial^2 v_h}{\partial x_i^2} + \frac{\partial \Delta v}{\partial x_i} \frac{\partial v_h}{\partial x_i}\Big) \\ &+ \sum_{1\le i\ne j\le n} \sum_{T\in\mathcal{T}_h} \int_T \Big(\frac{\partial^2 v}{\partial x_i \partial x_j} \frac{\partial^2 v_h}{\partial x_i \partial x_j} + \frac{\partial^3 v}{\partial x_i^2 \partial x_j} \frac{\partial v_h}{\partial x_j}\Big) \\ &- \sum_{1\le i\ne j\le n} \sum_{T\in\mathcal{T}_h} \int_T \Big(\frac{\partial^2 v}{\partial x_i^2} \frac{\partial^2 v_h}{\partial x_j^2} + \frac{\partial^3 v}{\partial x_i^2 \partial x_j} \frac{\partial v_h}{\partial x_j}\Big),\end{aligned} \tag{23}$$

we obtain the conclusion of the lemma.

Theorem 2 *Let V_{h0} be the finite element space corresponding to NZT element, and let u and u_h be the solutions of problems (3) and (5) respectively. Then*

$$\lim_{h\to 0} \|u - u_h\|_{2,h} = 0, \tag{24}$$

and there exists a constant C independent of h *such that*

$$\|u - u_h\|_{2,h} \le Ch|u|_{3,\Omega} \tag{25}$$

when $u \in H^3(\Omega)$.

Proof From Lemma 2 we see that NZT element passes the F-E-M-Test in [12]. Hence NZT element passes the generalized patch test. By Theorem 1 and the fact that $H_0^2(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in norm $\|\cdot\|_{2,\Omega}$, we obtain

$$\lim_{h\to 0} \inf_{v_h \in V_{h0}} \|v - v_h\|_{2,h} = 0, \quad \forall v \in H_0^2(\Omega).$$

Thus (24) is true by the result in [16].

By the generalized Poincare-Friedrichs inequality [17] and the Strang Lemma (see [6] or [15]), we have

$$\|u - u_h\|_{2,h} \le C \left(\inf_{w_h \in V_{h0}} \|u - w_h\|_{2,h} + \sup_{w_h \in V_{h0}\, w_h \neq 0} \frac{|\, a_h(u, w_h) - (f, w_h)|}{\|w_h\|_{2,h}} \right).$$

Then (25) follows from (13) and (21).

5 Concluding remarks

To construct a convergent Z-type nonconforming element for the fourth order elliptic boundary value problems, is motivated by the theoretical interest and the efficiency consideration in practical computation. In this paper, the NZT element, a new n-dimensional C^0 nonconforming simplex finite element, is constructed and analyzed. The NZT element uses the same degrees of freedom with the Zienkiewicz element and the different shape function space. Unlike the Zienkiewicz element, the NZT element is convergent and passes the patch test in general grids.

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