# A Robust Finite Element Method for 3-D Elliptic Singular Perturbation Problem \*

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**Abstract.** This paper proposes a robust finite element method for a three dimensional fourth order elliptic singular perturbation problem. The method uses the three dimensional Morley element and replaces the finite element functions in the part of bilinear form corresponding to the second order differential operator by a suitable approximation. To give such an approximation, the paper constructs a convergent nonconforming element for the second order problem. The paper shows that the method converges uniformly in the perturbation parameter.

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## 1. Introduction

Let  $\Omega$  be a bounded polyhedral domain of  $\mathbb{R}^n$  with  $1 \leq n \leq 3$ . Denote the boundary of  $\Omega$  by  $\partial\Omega$ . For  $f \in L^2(\Omega)$ , we consider the following boundary value problem of fourth order elliptic singular perturbation equation:

$$\begin{cases} \varepsilon^2 \Delta^2 u - \Delta u = f, & \text{in } \Omega, \\ u|_{\partial\Omega} = \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0 \end{cases}$$
(1.1)

where  $\nu = (\nu_1, \dots, \nu_n)^{\top}$  is the unit outer normal of  $\partial \Omega$ ,  $\Delta$  is the standard Laplacian operator and  $\varepsilon$  is a real small parameter with  $0 < \varepsilon \leq 1$ . When  $\varepsilon \to 0$  the differential equation formally degenerates to Poisson equation.

In two dimensional case, the Morley element was proposed in [9] for the plate bending problem. The Morley element is convergent for fourth order elliptic problem, but is divergent for second order problem (see [5], [8] and [13]). The Morley element and an  $C^0$  modified Morley element for problem (1.1) were discussed in [10]. It was was shown that the modified Morley element is uniformly convergent with respective to  $\varepsilon$  while the Morley element does not converges when  $\varepsilon \to 0$ . Two non  $C^0$  nonconforming elements were proposed in [4] by the double set parameter technique. These two elements were also proved to be uniformly

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convergent. A modified Morley element method for problem (1.1) was proposed in [15], it is convergent uniformly with respective to  $\varepsilon$ . This method also uses the Morley element (or the rectangle Morley element), but the linear approximation (or the bilinear approximation) of finite element functions is used in the part of the bilinear form corresponding to the second order differential term.

In this paper, we consider three dimensional case. The three dimensional Morley element can be found in [11] or in [14]. We will take a similar way used in [15] and propose a modified Morley element method for problem (1.1). We will use certain approximation of finite element functions in the part of the bilinear form corresponding to the second order differential term. It will be shown that the modified method converges uniformly in the perturbation parameter  $\varepsilon$ . The three dimensional Morley element uses the integral averages of function over all edges as degrees of freedom instead of the function values at vertices. To given a suitable approximation of finite element function, we need to construct a convergent nonconforming finite element for Poisson equation with the integral averages of function over all edges as degrees of freedom.

Problem (1.1) is a boundary value problem of a stationary linearizing form of the Cahn-Hilliard equation. The modelling in material science makes use of the Cahn-Hilliard equations in three dimensions (see [3, 2, 6]). Besides the theoretical interest, our new finite element method is hoped to be useful in the computation of the Cahn-Hilliard equation.

The paper is organized as follows. The rest of this section lists some preliminaries. Section 2 describes a nonconforming finite element for Poisson equation. Section 3 gives the detail descriptions of the modified Morley element method. Section 4 shows the uniform convergence of the method.

Throughout this paper, we assume n = 3. For nonnegative integer s, let  $H^s(\Omega)$ ,  $\|\cdot\|_{s,\Omega}$ and  $|\cdot|_{s,\Omega}$  denote the usual Sobolev space, norm and semi-norm respectively. Let  $H_0^s(\Omega)$  be the closure of  $C_0^{\infty}(\Omega)$  in  $H^s(\Omega)$  with respect to the norm  $\|\cdot\|_{s,\Omega}$  and  $(\cdot, \cdot)$  denote the inner product of  $L^2(\Omega)$ . Define

$$a(v,w) = \int_{\Omega} \sum_{i,j=1}^{3} \frac{\partial^2 v}{\partial x_i \partial x_j} \frac{\partial^2 w}{\partial x_i \partial x_j}, \quad \forall v, w \in H^2(\Omega),$$
(1.2)

$$b(v,w) = \int_{\Omega} \sum_{i=1}^{3} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_i}, \quad \forall v, w \in H^1(\Omega).$$
(1.3)

The weak form of problem (1.1) is: to find  $u \in H^2_0(\Omega)$  such that

$$\varepsilon^2 a(u,v) + b(u,v) = (f,v), \quad \forall v \in H^2_0(\Omega).$$
(1.4)

Let  $u^0$  be the solution of following boundary value problem:

$$\begin{cases} -\Delta u^0 = f, & \text{in } \Omega, \\ u^0|_{\partial\Omega} = 0 \end{cases}$$
(1.5)

For mesh size h, let  $\mathcal{T}_h$  be a triangulation of  $\Omega$  consisting of tetrahedrons. For each  $T \in \mathcal{T}_h$ , let  $h_T$  be the diameter of the smallest ball containing T and  $\rho_T$  be the diameter of the largest ball contained in T. Let  $\{\mathcal{T}_h\}$  be a family of triangulations with  $h \to 0$ . Throughout the paper, we assume that  $h_T \leq h \leq \eta \rho_T$ ,  $\forall T \in \mathcal{T}_h$ , with  $\eta$  a positive constant independent of h.

#### 2. A nonconforming element for Poisson equation

For a subset  $B \subset \mathbb{R}^3$  and a nonnegative integer r, let  $P_r(B)$  be the space of all polynomials with degree not greater than r.

Given a tetrahedron T, its four vertices is denoted by  $a_j$ ,  $1 \leq j \leq 4$ . The face of T opposite  $a_j$  is denoted by  $F_j$ ,  $1 \leq j \leq 4$ . The edge with  $a_i$  and  $a_j$  as its vertices, is denoted by  $S_{ij}$ ,  $1 \leq i < j \leq 4$ . Denote the measures of T,  $F_i$  and  $S_{ij}$  by |T|,  $|F_i|$  and  $|S_{ij}|$  respectively. Let  $\lambda_1, \dots, \lambda_4$  be the barycentric coordinates of T. Define

$$q_1 = (\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4), \quad q_2 = (\lambda_1 - \lambda_2)(\lambda_4 - \lambda_3)$$

We define a nonconforming element  $(T, P_T^s, \Phi_T^s)$  for Poisson equation by

- 1) T is a tetrahedron.
- **2)**  $P_T^s = P_1(T) + \operatorname{span}\{q_1, q_2\}.$
- **3)** For  $v \in C^0(T)$ ,

$$\Phi_T^s(v) = (\phi_{12}(v), \phi_{13}(v), \phi_{14}(v), \phi_{23}(v), \phi_{24}(v), \phi_{34}(v))^{\top}$$

with

$$\phi_{ij}(v) = \frac{1}{|S_{ij}|} \int_{S_{ij}} v, \ 1 \le i < j \le 4$$

For  $1 \le i < j \le 4$ , let  $1 \le k < l \le 4$  and  $\{k, l\} \cap \{i, j\} = \emptyset$ , and define

$$p_{ij} = \frac{2}{3}(\lambda_i + \lambda_j) - \frac{1}{3}(\lambda_k + \lambda_l) + 2\lambda_i\lambda_j + 2\lambda_k\lambda_l - \sum_{i_1=i,j}\sum_{i_2=k,l}\lambda_{i_1}\lambda_{i_2}.$$
 (2.1)

Set

$$\tilde{p}_{ij} = \frac{2}{3}(\lambda_i + \lambda_j) - \frac{1}{3}(\lambda_k + \lambda_l).$$

Then the following identities can be verified

$$\begin{cases} p_{12} = \tilde{p}_{12} + 2q_1 + q_2, \ p_{13} = \tilde{p}_{13} - q_1 - 2q_2, \ p_{14} = \tilde{p}_{14} - q_1 + q_2, \\ p_{23} = \tilde{p}_{23} - q_1 + q_2, \ p_{24} = \tilde{p}_{24} - q_1 - 2q_2, \ p_{34} = \tilde{p}_{34} + 2q_1 + q_2. \end{cases}$$
(2.2)

That is,  $p_{ij} \in P_T^s$ ,  $1 \le i < j \le 4$ . Denote by  $\delta_{ij}$  the Kronecker delta. By directly computing, we obtain

$$\frac{1}{|S_{kl}|} \int_{S_{kl}} p_{ij} = \delta_{ik} \delta_{jl}, \quad 1 \le i < j \le 4, \ 1 \le k < l \le 4.$$
(2.3)

Hence,  $p_{ij}$ ,  $1 \le i < j \le 4$ , are the basis functions corresponding to the degrees of freedom. This leads to that  $\Phi_T^s$  is  $P_T^s$ -unisolvent.

The interpolation operator  $\Pi_T^s$  corresponding to  $(T, P_T^s, \Phi_T^s)$  is written as

$$\Pi_T^s v = \sum_{1 \le i < j \le 4} p_{ij} \phi_{ij}(v), \quad \forall v \in C^0(T).$$
(2.4)

For  $v \in L^2(\Omega)$  and  $v|_T \in C^0(T), \forall T \in \mathcal{T}_h$ , define  $\Pi_h^s v$  by

$$\Pi_h^s v|_T = \Pi_T^s(v|_T), \quad \forall T \in \mathcal{T}_h.$$

$$(2.5)$$

By the interpolation theory (refer to [5]) we obtain the following lemma.

**Lemma 2.1** There exists a constant C independent of h such that

$$|v - \Pi_T^s v|_{m,T} \le Ch^{2-m} |v|_{2,T}, \quad 0 \le m \le 2, \ \forall v \in H^2(T)$$
(2.6)

is true for all  $T \in \mathcal{T}_h$ .

By a direct computation we have the following lemma.

Lemma 2.2 Given a tetrahedron T, the following equality is true,

$$\frac{1}{|F_i|} \int_{F_i} p = \frac{1}{9} \sum_{\substack{1 \le j \le k \le 4\\ j \ne i, k \ne i}} \frac{1}{|S_{jk}|} \int_{S_{jk}} p, \quad 1 \le i \le 4, \ \forall p \in P_T^s.$$
(2.7)

By the above two lemmas and the mathematical theory (refer to [8], [12] or [5]) we obtain that this element is convergent for the boundary value problem of three dimensional Poisson equation.

# 3. Modified Morley element method

The Morley element can be described by  $(T, P_T^M, \Phi_T^M)$  with

- 1) T is a tetrahedron.
- **2)**  $P_T^M = P_2(T).$
- 3)  $\Phi_T^M$  is the vector of degrees of freedom whose components are:

$$\frac{1}{|S_{ij}|} \int_{S_{ij}} v, \ 1 \le i < j \le 4; \quad \frac{1}{|F_j|} \int_{F_j} \frac{\partial v}{\partial \nu}, \ 1 \le j \le 4$$

for  $v \in C^1(T)$ .

For each  $\mathcal{T}_h$ , let  $V_h$  and  $V_{h0}$  be the corresponding finite element spaces associated with the Morley element for the discretization of  $H^2(\Omega)$  and  $H^2_0(\Omega)$  respectively. This defines two family of finite element spaces  $\{V_h\}$  and  $\{V_{h0}\}$ . It is known that  $V_h \not\subset H^2(\Omega)$  and  $V_{h0} \not\subset H^2_0(\Omega)$ . Let  $\Pi_h$  be the interpolation operator corresponding to the Morley element and  $\mathcal{T}_h$ .

We define, for  $v, w \in L^2(\Omega)$  and  $v|_T, w|_T \in H^2(T), \forall T \in \mathcal{T}_h$ ,

$$a_h(v,w) = \sum_{T \in \mathcal{T}_h} \int_T \sum_{i,j=1}^3 \frac{\partial^2 v}{\partial x_i \partial x_j} \frac{\partial^2 w}{\partial x_i \partial x_j},$$
(3.1)

$$b_h(v,w) = \sum_{T \in \mathcal{T}_h} \int_T \sum_{i=1}^3 \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_i}.$$
(3.2)

The standard finite element method for problem (1.4) corresponding to the Morley element is: to find  $u_h \in V_{h0}$  such that

$$\varepsilon^2 a_h(u_h, v_h) + b_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_{h0}.$$
(3.3)

We consider the following modified Morley element method: to find  $u_h \in V_{h0}$  such that

$$\varepsilon^2 a_h(u_h, v_h) + b_h(\Pi_h^s u_h, \Pi_h^s v_h) = (f, \Pi_h^s v_h), \quad \forall v_h \in V_{h0}.$$

$$(3.4)$$

Problem (3.4) has unique solution when  $\varepsilon > 0$ . When  $\varepsilon = 0$ , the problem degenerates to

$$b_h(\Pi_h^s u_h, \Pi_h^s v_h) = (f, \Pi_h^s v_h), \quad \forall v_h \in V_{h0}.$$
(3.5)

Although the solution of problem (3.5) is not unique yet,  $\Pi_h^s u_h$  is uniquely determined. Actually,  $\Pi_h^s u_h$  is the exact finite element solution of the element for problem (1.5) given in previous section.

Now we consider two examples. Let  $\Omega = [-1, 1]^3$  and

$$u_1(x) = (1 - x_1^2)^2 (1 - x_2^2)^2 (1 - x_3^2)^2,$$
  
$$u_2(x) = (1 + \cos \pi x_1)(1 + \cos \pi x_2)(1 + \cos \pi x_3).$$

Let  $i \in \{1, 2\}$ . For  $\varepsilon \ge 0$ , set  $f = \varepsilon^2 \Delta^2 u_i - \Delta u_i$ . Then  $u_i$  is the solution of problem (1.1) when  $\varepsilon > 0$ , and is the solution of problem (1.5) when  $\varepsilon = 0$ .

We first divide  $\Omega$  into 12 tetrahedral elements with h = 2 as shown in Figure 1, then use the global regular refinement strategy provided in [1] to get the mesh sequence.



Figure 1: The initial mesh

Define

$$|||v_h|||_{\varepsilon,h} = \left(\varepsilon^2 a_h(v_h, v_h) + b_h(\Pi_h^s v_h, \Pi_h^s v_h)\right)^{1/2}, \quad \forall v_h \in V_{h0}$$

Different values of  $\varepsilon$  and h are chosen to demonstrate the behaviors of the following relative error of the modified Morley element method,

$$E_{\varepsilon,h} = \frac{\||\Pi_h u - u_h\||_{\varepsilon,h}}{\||\Pi_h u\||_{\varepsilon,h}}$$
(3.6)

where  $u_h$  is the solution of problem (3.4).

Let  $g = \Delta^2 u_i$ , then  $u_i$  is the solution of the following boundary value problem of biharmonic equation,

$$\begin{cases} \Delta^2 u = g, & \text{in } \Omega, \\ u|_{\partial\Omega} = \frac{\partial u}{\partial\nu}\Big|_{\partial\Omega} = 0 \end{cases}$$
(3.7)

For comparison, we also consider the error of finite element solution to problem (3.7). Let  $\tilde{u}_h \in V_{h0}$  be the solution of the following problem,

$$a_h(\tilde{u}_h, v_h) = (g, \Pi_h^s v_h), \quad \forall v_h \in V_{h0}.$$

$$(3.8)$$

In this situation, the relative error  $\tilde{E}_h$  is presented by

$$\tilde{E}_{h}^{2} = \frac{a_{h}(\Pi_{h}u - \tilde{u}_{h}, \Pi_{h}u - \tilde{u}_{h})}{a_{h}(\Pi_{h}u, \Pi_{h}u)}$$
(3.9)

For the modified Morley element method in the case of  $f = \varepsilon^2 \Delta^2 u_1 - \Delta u_1$  and  $g = \Delta^2 u_1$ ,  $E_{\varepsilon,h}$  and  $\tilde{E}_h$ , corresponding some  $\varepsilon$  and h, are listed in Table 1. In the case that  $f = \varepsilon^2 \Delta^2 u_2 - \Delta u_2$  and  $g = \Delta^2 u_2$ ,  $E_{\varepsilon,h}$  and  $\tilde{E}_h$  are listed in Table 2. From Table 1 and Table 2 we see that the modified Morley element method converges

From Table 1 and Table 2 we see that the modified Morley element method converges for all  $\varepsilon \in [0, 1]$ . More precisely, the result shows that  $E_{\varepsilon,h}$  is linear with respect to h as well as  $E_{0,h}$  and  $\tilde{E}_h$  are.

$\epsilon ackslash h$	2	1	$2^{-1}$	$2^{-2}$	$2^{-3}$	$2^{-4}$
0	0.5800	0.2942	0.1654	0.08072	0.03969	0.01960
$2^{-10}$	0.5800	0.2942	0.1654	0.08071	0.03966	0.01958
$2^{-8}$	0.5802	0.2943	0.1654	0.0805	0.03923	0.01874
$2^{-6}$	0.5844	0.2950	0.1651	0.07802	0.03429	0.01276
$2^{-4}$	0.6492	0.3082	0.1680	0.06994	0.02814	0.01234
$2^{-2}$	1.438	0.5122	0.2923	0.1426	0.06951	0.03398
1	3.565	0.8335	0.4097	0.1959	0.09494	0.04634
$\infty$ (Biharmonic)	4.195	0.8872	0.4243	0.2021	0.09781	0.04773

Table 1

$\epsilon ackslash h$	2	1	$2^{-1}$	$2^{-2}$	$2^{-3}$	$2^{-4}$
0	0.7717	0.3048	0.1778	0.08484	0.04107	0.02009
$2^{-10}$	0.7717	0.3048	0.1778	0.08483	0.04105	0.02003
$2^{-8}$	0.7721	0.3049	0.1777	0.08466	0.04063	0.01920
$2^{-6}$	0.7776	0.3054	0.1777	0.08226	0.03570	0.01316
$2^{-4}$	0.8643	0.3140	0.1822	0.07345	0.02838	0.01209
$2^{-2}$	1.919	0.4598	0.2949	0.1401	0.06752	0.03288
1	4.788	0.7376	0.4012	0.1907	0.09203	0.04484
$\infty$ (Biharmonic)	5.646	0.7877	0.4144	0.1966	0.09480	0.04618

Table 2

### 4. Convergence analysis

In this section, we discuss the convergence properties of the modified Morley element methods given in previous section.

We introduce the following mesh dependent norm  $\|\cdot\|_{m,h}$  and semi-norm  $|\cdot|_{m,h}$ :

$$\|v\|_{m,h} = \left(\sum_{T \in \mathcal{T}_h} \|v\|_{m,T}^2\right)^{1/2}, \quad |v|_{m,h} = \left(\sum_{T \in \mathcal{T}_h} |v|_{m,T}^2\right)^{1/2},$$

for  $v \in L^2(\Omega)$  that  $v|_T \in H^m(T), \forall T \in \mathcal{T}_h$ .

Let u and  $u_h$  be the solutions of problem (1.4) and (3.4) respectively.

**Lemma 4.1** There exists a constant C independent of h and  $\varepsilon$  such that for any  $v_h \in V_{h0}$ , there exists  $w_h \in H_0^1(\Omega)$  satisfying

$$\|v_h - w_h\|_{0,\Omega} + h|v_h - w_h|_{1,h} \le Ch^2 |v_h|_{2,h},\tag{4.1}$$

$$\|\Pi_h^s v_h - w_h\|_{0,\Omega} + h |\Pi_h^s v_h - w_h|_{1,h} \le Ch |\Pi_h^s v_h|_{1,h}.$$
(4.2)

*Proof.* Let  $v_h \in V_{h0}$ . For  $T \in \mathcal{T}_h$ , denote by  $\Pi_T^1$  the linear interpolation operator with function values at all vertices of T as degrees of freedom. Define  $\Pi_h^1 v$  by

$$\Pi_h^1 v|_T = \Pi_T^1(v|_T), \quad \forall T \in \mathcal{T}_h$$

for function  $v \in L^2(\Omega)$  and  $v|_T \in C^0(T), \forall T \in \mathcal{T}_h$ . By the interpolation theory, the following inequality is true

$$|\Pi_h^s v_h - \Pi_h^1 \Pi_h^s v_h|_{m,h} \le C h^{2-m} |\Pi_h^s v_h|_{2,h}, \quad 0 \le m \le 1.$$
(4.3)

Given a set  $B \subset \mathbb{R}^n$ , let  $\mathcal{T}_h(B) = \{T \in \mathcal{T}_h \mid B \cap T \neq \emptyset\}$  and  $N_h(B)$  the number of the elements in  $\mathcal{T}_h(B)$ .

Now we define  $w_h \in H_0^1(\Omega)$  as follows: for any  $T \in \mathcal{T}_h$ 

- i)  $w_h|_T \in P_1(T)$ .
- ii) if vertex  $a_i$  of T is in  $\Omega$  then

$$w_h(a_i) = \frac{1}{N_h(a_i)} \sum_{T' \in \mathcal{T}_h(a_i)} (\Pi_h^s v_h|_{T'})(a_i).$$

Then  $w_h$  is well defined. We will show

$$|\Pi_h^s v_h - w_h|_{m,h} \le Ch^{2-m} |\Pi_h^s v_h|_{2,h}, \quad 0 \le m \le 1.$$
(4.4)

By the affine technique, we can show that

$$|p|_{m,T}^2 \le Ch^{3-2m} \sum_{i=1}^4 |p(a_i)|^2, \ \forall p \in P_1(T), \ m = 0, 1.$$
(4.5)

Set  $\varphi = \prod_{h=1}^{1} \prod_{h=1}^{s} v_h - w_h$  and  $\psi = \prod_{h=1}^{s} v_h$ . Obviously,  $\varphi|_T \in P_1(T)$ ,  $\forall T \in \mathcal{T}_h$ . For  $T \in \mathcal{T}_h$ , let  $\varphi_T = \varphi|_T$  and  $\psi_T = \psi|_T$ .

If vertex  $a_i$  of T is in  $\Omega$  then by the definition of  $w_h$ ,

$$\varphi(a_i) = \psi_T(a_i) - \frac{1}{N_h(a_i)} \sum_{T' \in \mathcal{T}_h(a_i)} \psi_{T'}(a_i)$$
$$= \frac{1}{N_h(a_i)} \sum_{T' \in \mathcal{T}_h(a_i)} \left( \psi_T(a_i) - \psi_{T'}(a_i) \right)$$

For  $T' \in \mathcal{T}_h(a_i)$  there exist  $T_1, \dots, T_J \in \mathcal{T}_h(a_i)$  such that  $T_1 = T, T_J = T'$  and  $\tilde{F}_j = T_j \cap T_{j+1}$ is a common face of  $T_j$  and  $T_{j+1}$  and  $a_i \in \tilde{F}_j, 1 \leq j < J$ . By the inverse inequality, we have

$$\begin{aligned} \left|\psi_{T}(a_{i})-\psi_{T'}(a_{i})\right|^{2} &= \Big|\sum_{j=1}^{J-1} \left(\psi_{T_{j}}(a_{i})-\psi_{T_{j+1}}(a_{i})\right)\Big|^{2} \\ &\leq C\sum_{j=1}^{J-1} \left|\psi_{T_{j}}(a_{i})-\psi_{T_{j+1}}(a_{i})\right|^{2} \leq Ch^{-2}\sum_{j=1}^{J-1} \left|\psi_{T_{j}}-\psi_{T_{j+1}}\right|^{2}_{0,\tilde{F}_{j}}.\end{aligned}$$

On each edge of  $\tilde{F}_j$ , the integral average of  $\psi_{T_j}$  is equal to the one of  $\psi_{T_{j+1}}$  by the definition of  $\psi$ . Hence

$$\left|\psi_{T_j} - \psi_{T_{j+1}}\right|_{0,\tilde{F}_j}^2 \le Ch^3 \left(|\psi|_{2,T_j}^2 + |\psi|_{2,T_{j+1}}^2\right)$$

Then

$$\left|\psi_T(a_i) - \psi_{T'}(a_i)\right|^2 \le Ch \sum_{j=1}^J |\psi|_{2,T_j}^2$$

Since  $N_h(T)$  is bounded, we get

$$|\varphi(a_i)|^2 \le Ch \sum_{T' \in \mathcal{T}_h(T)} |\psi|^2_{2,T'}$$
(4.6)

If vertex  $a_i$  of T is on  $\partial \Omega$  then exists  $T' \in \mathcal{T}_h(a_i)$  with a face F of T' belonging to  $\partial \Omega$ and  $a_i \in F$ . By the definitions of  $w_h$ ,

$$|\varphi(a_i)| = |\psi_T(a_i) - \psi_{T'}(a_i) + \psi_{T'}(a_i)| \le |\psi_T(a_i) - \psi_{T'}(a_i)| + |\psi_{T'}(a_i)|$$

Since the integral average of  $\psi_{T'}$  on each edge of F vanishes,

$$|\psi_{T'}(a_i)|^2 \le Ch^{-2} |\psi_{T'}|^2_{0,F} \le Ch |\psi|^2_{2,T'}$$

by the inverse inequality. By similar analysis for  $|\psi_T(a_i) - \psi_{T'}(a_i)|$ , we conclude that (4.6) is also true in this case.

Combining (4.5) and (4.6), we obtain

$$h^{2m} |\varphi|_{m,T}^2 \le Ch^4 \sum_{T' \in \mathcal{T}_h(T)} |\psi|_{2,T'}^2.$$

Summing the above inequality over all  $T \in \mathcal{T}_h$ , we get

$$h^{2m}|\varphi|^2_{m,h} \leq Ch^4 \sum_{T \in \mathcal{T}_h} \sum_{T' \in \mathcal{T}_h(T)} |\psi|^2_{2,T'}.$$

Consequently,

$$h^{2m} |\varphi|_{m,h}^2 \le Ch^4 |\psi|_{2,h}^2.$$
(4.7)

Inequality (4.4) follows from (4.7) and (4.3).

We obtain (4.2) by (4.4) and the inverse inequality, and (4.1) by (4.4) and Lemma 2.1.  $\Box$ 

**Lemma 4.2** There exists a constant C independent of h and  $\varepsilon$  such that for any  $v_h \in V_{h0}$ 

$$|b_h(\Pi_h^s u, \Pi_h^s v_h) + (\Delta u, \Pi_h^s v_h)| \le Ch |u|_{2,\Omega} |\Pi_h^s v_h|_{1,h},$$
(4.8)

$$|a_h(u, v_h) - (\Delta^2 u, \Pi_h^s v_h)| \le C(h|u|_{3,\Omega} + h^2 ||\Delta^2 u||_{0,\Omega}) |v_h|_{2,h},$$
(4.9)

when  $u \in H^3(\Omega)$ .

*Proof.* Let  $v_h \in V_{h0}$ . By Green's formula

$$b_h(\Pi_h^s u, \Pi_h^s v_h) + (\Delta u, \Pi_h^s v_h) = b_h(\Pi_h^s u - u, \Pi_h^s v_h) + \sum_{T \in \mathcal{T}_h} \int_{\partial T} \frac{\partial u}{\partial \nu} \Pi_h^s v_h.$$

Given  $T \in \mathcal{T}_h$  and a face F of T, let  $P_F^0$  be the orthogonal projection operator from  $L^2(F)$  to  $P_0(F)$ . By Lemma 2.2, we have

$$\sum_{T \in \mathcal{T}_h} \int_{\partial T} \frac{\partial u}{\partial \nu} \Pi_h^s v_h = \sum_{T \in \mathcal{T}_h} \sum_{F \subset \partial T} \int_F \left( \frac{\partial u}{\partial \nu} - P_F^0 \frac{\partial u}{\partial \nu} \right) (\Pi_h^s v_h - P_F^0 \Pi_h^s v_h).$$

By the interpolation theory and the Schwarz inequality we obtain

$$\left|\sum_{T\in\mathcal{T}_{h}}\int_{\partial T}\frac{\partial u}{\partial\nu}\Pi_{h}^{s}v_{h}\right| \leq Ch|u|_{2,\Omega}|\Pi_{h}^{s}v_{h}|_{1,h}.$$
(4.10)

On the other hand,

$$|b_h(\Pi_h^s u - u, \Pi_h^s v_h)| \le Ch|u|_{2,\Omega}|\Pi_h^s v_h|_{1,h}.$$

Hence (4.8) follows.

Now let  $\phi \in H^1(\Omega)$ . Let  $i, j \in \{1, 2, 3\}$ . It is known that the integral average of  $\frac{\partial}{\partial x_j} v_h$ on F is continuous through F and vanishes when  $F \subset \partial \Omega$ . Then Green's formula gives

$$\sum_{T \in \mathcal{T}_h} \int_T \left( \phi \frac{\partial^2 v_h}{\partial x_i \partial x_j} + \frac{\partial \phi}{\partial x_i} \frac{\partial v_h}{\partial x_j} \right)$$
$$= \sum_{T \in \mathcal{T}_h} \int_{\partial T} \phi \frac{\partial v_h}{\partial x_j} \nu_i = \sum_{T \in \mathcal{T}_h} \sum_{F \subset \partial T} \int_F \phi \frac{\partial v_h}{\partial x_j} \nu_i$$
$$= \sum_{T \in \mathcal{T}_h} \sum_{F \subset \partial T} \int_F \phi \left( \frac{\partial v_h}{\partial x_j} - P_F^0 \frac{\partial v_h}{\partial x_j} \right) \nu_i$$
$$= \sum_{T \in \mathcal{T}_h} \sum_{F \subset \partial T} \int_F (\phi - P_F^0 \phi) \left( \frac{\partial v_h}{\partial x_j} - P_F^0 \frac{\partial v_h}{\partial x_j} \right) \nu_i$$

From the Schwarz inequality and the interpolation theory we obtain

$$\begin{split} \sum_{T \in \mathcal{T}_h} \sum_{F \subset \partial T} \int_F (\phi - P_F^0 \phi) \Big( \frac{\partial v_h}{\partial x_j} - P_F^0 \frac{\partial v_h}{\partial x_j} \Big) \nu_i \Big| \\ & \leq \sum_{T \in \mathcal{T}_h} \sum_{F \subset \partial T} \|\phi - P_F^0 \phi\|_{0,F} \Big\| \frac{\partial v_h}{\partial x_j} - P_F^0 \frac{\partial v_h}{\partial x_j} \Big\|_{0,F} \\ & \leq C \sum_{T \in \mathcal{T}_h} h |\phi|_{1,T} |v_h|_{2,T} \leq Ch |\phi|_{1,\Omega} |v_h|_{2,h}. \end{split}$$

Consequently, we obtain that for any  $\phi \in H^1(\Omega)$ ,  $v_h \in V_{h0}$ ,  $i, j \in \{1, 2, 3\}$ ,

$$\Big|\sum_{T\in\mathcal{T}_h}\int_T \Big(\phi\frac{\partial^2 v_h}{\partial x_i\partial x_j} + \frac{\partial\phi}{\partial x_i}\frac{\partial v_h}{\partial x_j}\Big)\Big| \le Ch|\phi|_{1,\Omega}|v_h|_{2,h}.$$
(4.11)

Let  $w_h \in H_0^1(\Omega)$  be as in (4.1) and (4.2). Then

$$a_{h}(u, v_{h}) - (\Delta^{2}u, \Pi_{h}^{s}v_{h}) = (\Delta^{2}u, w_{h} - \Pi_{h}^{s}v_{h})$$

$$+ \sum_{i=1}^{3} \sum_{T \in \mathcal{T}_{h}} \int_{T} \frac{\partial \Delta u}{\partial x_{i}} \frac{\partial (w_{h} - v_{h})}{\partial x_{i}}$$

$$+ \sum_{i=1}^{3} \sum_{T \in \mathcal{T}_{h}} \int_{T} \left( \Delta u \frac{\partial^{2}v_{h}}{\partial x_{i}^{2}} + \frac{\partial \Delta u}{\partial x_{i}} \frac{\partial v_{h}}{\partial x_{i}} \right)$$

$$+ \sum_{1 \leq i \neq j \leq 3} \sum_{T \in \mathcal{T}_{h}} \int_{T} \left( \frac{\partial^{2}u}{\partial x_{i} \partial x_{j}} \frac{\partial^{2}v_{h}}{\partial x_{i} \partial x_{j}} + \frac{\partial^{3}u}{\partial x_{i}^{2} \partial x_{j}} \frac{\partial v_{h}}{\partial x_{j}} \right)$$

$$- \sum_{1 \leq i \neq j \leq 3} \sum_{T \in \mathcal{T}_{h}} \int_{T} \left( \frac{\partial^{2}u}{\partial x_{i}^{2}} \frac{\partial^{2}v_{h}}{\partial x_{j}^{2}} + \frac{\partial^{3}u}{\partial x_{i}^{2} \partial x_{j}} \frac{\partial v_{h}}{\partial x_{j}} \right).$$

$$(4.12)$$

We obtain (4.9) from (4.12), (4.11), (4.1) and Lemma 2.1.

**Theorem 4.1** There exists a constant C independent of h and  $\varepsilon$  such that

$$\varepsilon \|u - u_h\|_{2,h} + \|u - \Pi_h^s u_h\|_{1,h} \le Ch(|u|_{2,\Omega} + \varepsilon |u|_{3,\Omega} + \varepsilon h \|\Delta^2 u\|_{0,\Omega})$$
(4.13)

when  $u \in H^3(\Omega)$ .

*Proof.* Let  $\varphi_h = \prod_h u$ , then

$$\varepsilon \|u - u_h\|_{2,h} + \|u - \Pi_h^s u_h\|_{1,h} \le \varepsilon \|u - \varphi_h\|_{2,h} + \|u - \Pi_h^s \varphi_h\|_{1,h} + \varepsilon \|u_h - \varphi_h\|_{2,h} + \|\Pi_h^s (u_h - \varphi_h)\|_{1,h}.$$
(4.14)

Set  $v_h = u_h - \varphi_h$ . From (3.4) and (1.1), we derive that

$$\varepsilon^2 a_h(v_h, v_h) + b_h(\Pi_h^s v_h, \Pi_h^s v_h)$$

$$=\varepsilon^{2}a_{h}(u-\varphi_{h},v_{h})+b_{h}(\Pi_{h}^{s}(u-\varphi_{h}),\Pi_{h}^{s}v_{h})$$
$$+\varepsilon^{2}\Big((\Delta^{2}u,\Pi_{h}^{s}v_{h})-a_{h}(u,v_{h})\Big)$$
$$-\Big((\Delta u,\Pi_{h}^{s}v_{h})+b_{h}(\Pi_{h}^{s}u,\Pi_{h}^{s}v_{h})\Big).$$

By the interpolation theory, Lemma 2.1, (4.8) and (4.9), we have

$$\varepsilon^{2}a_{h}(v_{h},v_{h})+b_{h}(\Pi_{h}^{s}v_{h},\Pi_{h}^{s}v_{h})$$

$$\leq Ch(|u|_{2,\Omega}+\varepsilon|u|_{3,\Omega}+\varepsilon h\|\Delta^{2}u\|_{0,\Omega})(\varepsilon|v_{h}|_{2,h}+|\Pi_{h}^{s}v_{h}|_{1,h}).$$

Since

$$\varepsilon^{2} \|v_{h}\|_{2,h}^{2} + \|\Pi_{h}^{s}v_{h}\|_{1,h}^{2} \le C \Big(\varepsilon^{2}a_{h}(v_{h},v_{h}) + b_{h}(\Pi_{h}^{s}v_{h},\Pi_{h}^{s}v_{h})\Big)$$

we obtain that

$$\varepsilon \|u_h - \varphi_h\|_{2,h} + \|\Pi_h^s(u_h - \varphi_h)\|_{1,h} \le Ch\big(|u|_{2,\Omega} + \varepsilon |u|_{3,\Omega} + \varepsilon h \|\Delta^2 u\|_{0,\Omega}\big).$$
(4.15)

The theorem follows from the interpolation theory, (4.14) and (4.15). Similar to Lemma 5.1 in [10], we can prove the following lemma.

**Lemma 4.3** If  $\Omega$  is convex, then there exists a constant C independent of  $\varepsilon$  such that

$$\varepsilon^{-1/2} |u - u^0|_{1,\Omega} + \varepsilon^{1/2} |u|_{2,\Omega} + \varepsilon^{3/2} |u|_{3,\Omega} \le C ||f||_{0,\Omega}$$
(4.16)

for all  $f \in L^2(\Omega)$ .

**Lemma 4.4** There exists a constant C independent of  $\varepsilon$  and h such that

$$\|v\|_{0,\partial T} \le C \Big( h^{-1/2} \|v\|_{0,T} + \|v\|_{0,T}^{1/2} \|v\|_{1,T}^{1/2} \Big), \tag{4.17}$$

$$\sum_{F \subset \partial T} \|v - P_F^0 v\|_{0,F} \le C \|v\|_{0,T}^{1/2} |v|_{1,T}^{1/2},$$
(4.18)

for all  $v \in H^1(T)$  and  $T \in \mathcal{T}_h$ .

*Proof.* Let  $\hat{T}$  be the reference tetrahedron. From [7] we know that

$$\|\hat{v}\|_{0,\partial\hat{T}} \le C \|\hat{v}\|_{0,\hat{T}}^{1/2} \|\hat{v}\|_{1,\hat{T}}^{1/2}, \quad \forall \hat{v} \in H^1(\hat{T}).$$

$$(4.19)$$

Then we obtain (4.17) by the affine technique.

Now let  $T \in \mathcal{T}_h$  and let  $P_T^0$  be the orthogonal projection operator from  $L^2(T)$  to  $P_0(T)$ . For each  $\hat{F} \subset \partial \hat{T}$  and  $\hat{v} \in H^1(\hat{T})$ , we have by (4.19) and the interpolation theory,

$$\begin{split} \|\hat{v} - P_{\hat{F}}^{0}\hat{v}\|_{0,\hat{F}} &\leq \|\hat{v} - P_{\hat{T}}^{0}\hat{v} - P_{\hat{F}}^{0}(\hat{v} - P_{\hat{T}}^{0}\hat{v})\|_{0,\hat{F}} \\ &\leq C \|\hat{v} - P_{\hat{T}}^{0}\hat{v}\|_{0,\hat{T}}^{1/2} \|\hat{v} - P_{\hat{T}}^{0}\hat{v}\|_{1,\hat{T}}^{1/2} \leq \|\hat{v}\|_{0,\hat{T}}^{1/2} |\hat{v}|_{1,\hat{T}}^{1/2}. \end{split}$$

Consequently, we obtain (4.18) by the affine technique.

**Theorem 4.2** If  $\Omega$  is convex, then there exists a constant C independent of h and  $\varepsilon$  such that

$$\varepsilon \|u - u_h\|_{2,h} + \|u - \Pi_h^s u_h\|_{1,h} \le Ch^{1/2} \|f\|_{0,\Omega}.$$
(4.20)

*Proof.* From the interpolation theory, it is true that

$$||u - \Pi_h u||_{2,h}^2 \le C|u|_{2,\Omega} ||u - \Pi_h u||_{2,h} \le Ch|u|_{2,\Omega} |u|_{3,\Omega}.$$

By Lemma 4.3, we have

$$\varepsilon \| u - \Pi_h u \|_{2,h} \le C h^{1/2} \| f \|_{0,\Omega}.$$
(4.21)

Similar to (4.4) in [10], we can show that

$$\|v - \Pi_h^s v\|_{1,h}^2 \le Ch |v|_{1,\Omega} |v|_{2,\Omega}, \quad \forall v \in H_0^2(\Omega).$$
(4.22)

Using (4.22), we obtain

$$||u - u^{0} - \Pi_{h}^{s}(u - u^{0})||_{1,h}^{2} \le Ch|u - u^{0}|_{1,\Omega}|u - u^{0}|_{2,\Omega},$$

and we have, by the interpolation theory,

$$||u^0 - \Pi_h^s u^0||_{1,h} \le Ch |u^0|_{2,\Omega}$$

By Lemma 4.3 and the following inequalities

$$\|u^0\|_{2,\Omega} \le C \|f\|_{0,\Omega},\tag{4.23}$$

$$||u - \Pi_h^s u||_{1,h} \le ||u - u^0 - \Pi_h^s (u - u^0)||_{1,h} + ||u^0 - \Pi_h^s u^0||_{1,h},$$

we have

$$\|u - \Pi_h^s u\|_{1,h} \le Ch^{1/2} \|f\|_{0,\Omega}.$$
(4.24)

Set  $v_h = u_h - \prod_h u$ . Lemma 2.2 and Green's formula give

$$b_{h}(\Pi_{h}^{s}u, \Pi_{h}^{s}v_{h}) + (\Delta u, \Pi_{h}^{s}v_{h}) = b_{h}(\Pi_{h}^{s}u - u, \Pi_{h}^{s}v_{h})$$
$$+ \sum_{T \in \mathcal{T}_{h}} \sum_{F \subset \partial T} \int_{F} \left(\frac{\partial(u - u^{0})}{\partial\nu} - P_{F}^{0}\frac{\partial(u - u^{0})}{\partial\nu}\right)(\Pi_{h}^{s}v_{h} - P_{F}^{0}\Pi_{h}^{s}v_{h})$$
$$+ \sum_{T \in \mathcal{T}_{h}} \sum_{F \subset \partial T} \int_{F} \left(\frac{\partial u^{0}}{\partial\nu} - P_{F}^{0}\frac{\partial u^{0}}{\partial\nu}\right)(\Pi_{h}^{s}v_{h} - P_{F}^{0}\Pi_{h}^{s}v_{h}).$$

By the Schwarz inequality and the interpolation theory, we have

$$\begin{split} |b_h(\Pi_h^s u, \Pi_h^s v_h) + (\Delta u, \Pi_h^s v_h)| \\ &\leq C \sum_{T \in \mathcal{T}_h} \Big( |u - \Pi_h^s u|_{1,T} + h|u^0|_{2,T} \\ &+ h^{1/2} \sum_{F \subset \partial T} \Big| \frac{\partial (u - u^0)}{\partial \nu} - P_F^0 \frac{\partial (u - u^0)}{\partial \nu} \Big|_{0,F} \Big) |\Pi_h^s v_h|_{1,T} \end{split}$$

Then we obtain by (4.24), (4.18), (4.23) and Lemma 4.3 that,

$$|b_h(\Pi_h^s u, \Pi_h^s v_h) + (\Delta u, \Pi_h^s v_h)| \le Ch^{1/2} ||f||_{0,\Omega} ||\Pi_h^s v_h||_{1,h}.$$
(4.25)

Now let  $\phi \in H^1(\Omega)$  and  $i, j \in \{1, 2\}$ . From the proof of Lemma 4.2, we have

$$\begin{split} \Big| \sum_{T \in \mathcal{T}_h} \int_T \Big( \phi \frac{\partial^2 v_h}{\partial x_i \partial x_j} + \frac{\partial \phi}{\partial x_i} \frac{\partial v_h}{\partial x_j} \Big) \Big| \\ & \leq \sum_{T \in \mathcal{T}_h} \sum_{F \subset \partial T} \| \phi - P_F^0 \phi \|_{0,F} \Big\| \frac{\partial v_h}{\partial x_j} - P_F^0 \frac{\partial v_h}{\partial x_j} \Big\|_{0,F}. \end{split}$$

By the interpolation theory and (4.17), we have

$$\left|\sum_{T\in\mathcal{T}_{h}}\int_{T}\left(\phi\frac{\partial^{2}v_{h}}{\partial x_{i}\partial x_{j}}+\frac{\partial\phi}{\partial x_{i}}\frac{\partial v_{h}}{\partial x_{j}}\right)\right|\leq Ch^{1/2}\|\phi\|_{0,\Omega}^{1/2}\|\phi\|_{1,\Omega}^{1/2}|v_{h}|_{2,h}.$$
(4.26)

Let  $w_h \in H^1_0(\Omega)$  such that (4.1) and (4.2) are true. If  $\varepsilon \leq h$ , then by Green's formula we get

$$\sum_{T \in \mathcal{T}_h} \int_T \frac{\partial \phi}{\partial x_i} \frac{\partial (w_h - v_h)}{\partial x_i} = \sum_{T \in \mathcal{T}_h} \int_{\partial T} \phi \frac{\partial (w_h - v_h)}{\partial x_i} \nu_i$$
$$- \sum_{T \in \mathcal{T}_h} \int_T \phi \frac{\partial^2 (w_h - v_h)}{\partial x_i^2}.$$

By the Schwarz inequality, (4.1) and (4.17), we obtain

$$\begin{split} \Big| \sum_{T \in \mathcal{T}_h} \int_T \frac{\partial \phi}{\partial x_i} \frac{\partial (w_h - v_h)}{\partial x_i} \Big| &\leq \sum_{T \in \mathcal{T}_h} \|\phi\|_{0,\partial T} \Big\| \frac{\partial (w_h - v_h)}{\partial x_i} \Big\|_{0,\partial T} \\ &+ \sum_{T \in \mathcal{T}_h} \|\phi\|_{0,T} |w_h - v_h|_{2,T} \\ &\leq C \Big( h^{1/2} \|\phi\|_{0,\Omega}^{1/2} \|\phi\|_{1,\Omega}^{1/2} + \|\phi\|_{0,\Omega} \Big) |v_h|_{2,h}. \end{split}$$

Hence when  $\varepsilon \leq h$ 

$$\varepsilon^{2} \Big| \sum_{T \in \mathcal{T}_{h}} \int_{T} \frac{\partial \phi}{\partial x_{i}} \frac{\partial (w_{h} - v_{h})}{\partial x_{i}} \Big| \le Ch^{1/2} \Big( \varepsilon^{2} \|\phi\|_{0,\Omega}^{1/2} \|\phi\|_{1,\Omega}^{1/2} + \varepsilon^{3/2} \|\phi\|_{0,\Omega} \Big) |v_{h}|_{2,h}.$$
(4.27)

When  $\varepsilon > h$ , by the Schwarz inequality and (4.1) we have,

$$\varepsilon^{2} \Big| \sum_{T \in \mathcal{T}_{h}} \int_{T} \frac{\partial \phi}{\partial x_{i}} \frac{\partial (w_{h} - v_{h})}{\partial x_{i}} \Big| \le Ch \varepsilon^{2} |\phi|_{1,\Omega} |v_{h}|_{2,h} \le Ch^{1/2} \varepsilon^{5/2} |\phi|_{1,\Omega} |v_{h}|_{2,h}.$$
(4.28)

From (1.1) and (1.5) it follows that

$$\varepsilon^{2}(\Delta^{2}u, w_{h} - \Pi_{h}^{s}v_{h}) = (\Delta(u - u^{0}), w_{h} - \Pi_{h}^{s}v_{h}).$$
(4.29)

When  $\varepsilon > h$ , we have by (4.2) and Lemma 2.2

$$|(\Delta(u-u^0), w_h - \Pi_h^s v_h)| \le Ch|u-u^0|_{2,\Omega}|\Pi_h^s v_h|_{1,h} \le Ch^{1/2}\varepsilon^{1/2}|u-u^0|_{2,\Omega}|\Pi_h^s v_h|_{1,h}.$$

By Lemma 4.3 and (4.23) we get that

$$|\varepsilon^{2}(\Delta^{2}u, w_{h} - \Pi_{h}^{s}v_{h})| \le Ch^{1/2} ||f||_{0,\Omega} |\Pi_{h}^{s}v_{h}|_{1,h}$$
(4.30)

is true when  $\varepsilon > h$ .

On the other hand, we have

$$(\Delta(u-u^0), w_h - \Pi_h^s v_h) = \sum_{j=1}^3 \sum_{T \in \mathcal{T}_h} \left( \int_{\partial T} \frac{\partial(u-u^0)}{\partial x_j} (w_h - \Pi_h^s v_h) \nu_j - \int_T \frac{\partial(u-u^0)}{\partial x_j} \frac{\partial(w_h - \Pi_h^s v_h)}{\partial x_j} \right)$$

Then

$$|(\Delta(u-u^{0}), w_{h} - \Pi_{h}^{s}v_{h})| \leq \sum_{j=1}^{3} \sum_{T \in \mathcal{T}_{h}} \left( \left\| \frac{\partial(u-u^{0})}{\partial x_{j}} \right\|_{0,\partial T} \|w_{h} - \Pi_{h}^{s}v_{h}\|_{0,\partial T} + \|u-u^{0}\|_{1,T} \|w_{h} - \Pi_{h}^{s}v_{h}\|_{1,T} \right).$$

By (4.17), (4.2) and the Schwarz inequality, we obtain

$$\begin{aligned} |(\Delta(u-u^0), w_h - \Pi_h^s v_h)| \\ &\leq C \Big( h^{1/2} \|u - u^0\|_{1,\Omega}^{1/2} \|u - u^0\|_{2,\Omega}^{1/2} + \|u - u^0\|_{1,\Omega} \Big) |\Pi_h^s v_h|_{1,h}. \end{aligned}$$

From Lemma 4.3 and (4.29) we get

$$|\varepsilon^{2}(\Delta^{2}u, w_{h} - \Pi_{h}^{s}v_{h})| \leq C(h^{1/2} + \varepsilon^{1/2}) ||f||_{0,\Omega} |\Pi_{h}^{s}v_{h}|_{1,h}$$

That is, (4.30) is also true when  $\varepsilon \leq h$ .

From Lemma 4.3, (4.12), (4.26), (4.27), (4.28) and (4.30) we obtain

$$\varepsilon^{2}|a_{h}(u,v_{h}) - (\Delta^{2}u,\Pi_{h}^{s}v_{h})| \le Ch^{1/2}||f||_{0,\Omega} (\varepsilon|v_{h}|_{2,h} + |\Pi_{h}^{s}v_{h}|_{1,h}).$$
(4.31)

Combining (4.21), (4.24), (4.25), (4.31) and the proof of Theorem 4.1, we obtain the theorem.  $\hfill \Box$ 

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