Numerische Mathematik manuscript No.
(will be inserted by the editor)

The Morley Element for Fourth Order Elliptic Equations in Any Dimensions

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Received: November 9, 2004

Summary In this paper, the well-known nonconforming Morley element for biharmonic equations in two spatial dimensions is extended to any higher dimensions in a canonical fashion. The general n-dimensional Morley element consists of all quadratic polynomials defined on each n-simplex with degrees of freedom given by the integral average of the normal derivative on each $(n-1)$-subsimplex and the integral average of the function value on each $(n-2)$-subsimplex. Explicit expressions of nodal basis functions are also obtained for this element on general n-simplicial grids. Convergence analysis is given for this element when it is applied as a nonconforming finite element discretization for the biharmonic equation.

Key words Nonconforming finite element, forth order elliptic equation, biharmonic, Morley element.

1 Introduction

In this paper we consider nonconforming finite elements for higher dimensional fourth order elliptic equations. There are some well-known nonconforming finite elements in two dimensional case (cf. [1]-[4]). Among them, the Morley element is perhaps the most interesting one. The Morley element has the least number of degrees of freedom on each element for fourth order boundary value problems as its shape function space consists of only quadratic polynomials.

Motivated by both theoretical and practical interests, in our recent paper [9], we proposed and analyzed several tetrahedron nonconforming finite elements for three dimensional fourth order elliptic partial differential operators. But the extension of the Morley

⋆ The work was supported by the National Natural Science Foundation of China (10571006)
⋆⋆ This work was supported in part by NSF DMS-0209497 and NSF DMS-0215392 and the Changjiang Professorship through Peking University

element to three dimensions was then not obvious. In [5] an extension of the Morley element to n-dimensional case ($n > 2$) was given in a special manner, but it is interesting to note that this extension does not recover the two dimensional Morley element in the generalized family. In this paper, we generalize the two dimensional Morley element to any n-dimensional case ($n \geq 2$) in a more canonical fashion . Our generalization naturally recovers the two dimensional Morley element and also the three dimensional element given in [5]. Our new element is different from the element in [5] when $n > 3$. An error estimate was given in [5] for (and only for) the three dimensional case, but this estimate is not as sharp as the one that is obtained in this paper for any $n \geq 2$ in a unified analysis.

With quadratic polynomial as shape function on a general n-simplex, the degrees of freedom of the general Morley element presented in this paper are given by the integral average of the normal derivative on each $(n-1)$-subsimplex and the integral average of the function value on each $(n-2)$-subsimplex. It is intriguing that everything just fits together very nicely.

The paper is organized as follows. The rest of this section gives some notation. Section 2 describes the Morley element for the n-dimensional case with $n \geq 2$. Section 3 shows the convergence of the element (following the work of Shi [6]). The final section contains some brief concluding remarks.

We will use the following standard notation. Ω denotes a general bounded polyhedral domain in R^n ($n \geq 2$), $\partial\Omega$ the boundary of Ω, and $\nu = (\nu_1, \nu_2, \cdots, \nu_n)^\top$ the unit outer normal to $\partial\Omega$. For a nonnegative integer s, $H^s(\Omega)$, $\|\cdot\|_{s,\Omega}$ and $|\cdot|_{s,\Omega}$ denote the usual Sobolev space, its corresponding norm and semi-norm respectively, $H_0^s(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$ with respect to the norm $\|\cdot\|_{s,\Omega}$, and $(\cdot,\cdot)$ the inner product of $L^2(\Omega)$. For a subset $B \subset R^n$ and a nonnegative integer r, $P_r(B)$ denotes the space of all polynomials on B with degree not greater than r.

2 The n-Dimensional Morley element

In this section, we will give a detailed description of our new n-dimensional Morley element and discuss some basic properties. In §2.1, we will give the definition of the element and its justification. In §2.2, we will give an explicit construction of the nodal basis functions. In §2.3, we will discuss some basic properties.

2.1 The definition of the new element

Let T be a general n-simplex with $n+1$ vertices denoted by $a_i = (x_{1i}, x_{2i}, \cdots, x_{ni})^\top$ ($1 \leq i \leq n+1$) and with its barycentric coordinates denoted by $\lambda_1, \lambda_2, \cdots, \lambda_{n+1}$. We will use F_i ($1 \leq i \leq n+1$) to denote the $(n-1)$-dimensional subsimplex of T without a_i as its vertices and b_i its barycenter and S_{ij} ($1 \leq i < j \leq n+1$) to denote the $(n-2)$-dimensional subsimplex without a_i and a_j as its vertices. As usual, $|T|$, $|F_i|$ and $|S_{ij}|$ denote the measures of T, F_i and S_{ij} respectively.

Definition 1 (The n-dimensional Morley element) *The Morley element of n-dimension is defined by (T, P_T, Φ_T) with*

1. T is an n-simplex.
2. $P_T = P_2(T)$, the space of all quadratic polynomials.
3. Φ_T is the vector with its components the following degrees of freedom,

$$\frac{1}{|S_{ij}|}\int_{S_{ij}} v,\ 1 \le i < j \le n+1,\ \ \frac{1}{|F_j|}\int_{F_j} \frac{\partial v}{\partial \nu},\ 1 \le j \le n+1,\ \forall v \in C^1(T). \quad (1)$$

Remark 1. For $n = 2$, $S_{ij} = a_k$ is a vertex of T. We have

$$\frac{1}{|S_{ij}|}\int_{S_{ij}} v = v(a_k).$$

We thus recover the definition of the Morley element in two dimensions, see Fig. 1.

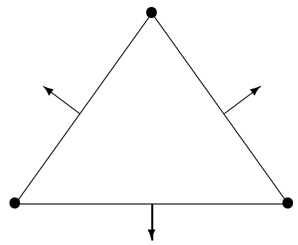

Fig. 1

Remark 2. For $n = 3$, S_{ij} are edges of the simplex, the degrees of freedom are illustrated in Fig. 2.

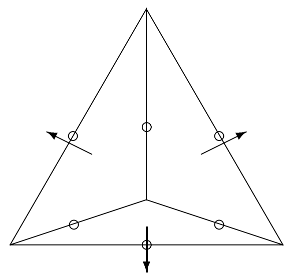

Fig. 2

Remark 3. Just like the 2-dimensional Morley element, the degrees of freedom of the element can be replaced by

$$\frac{1}{|S_{ij}|}\int_{S_{ij}} v,\ 1 \le i < j \le n+1,\ \frac{\partial v}{\partial \nu}(b_j),\ 1 \le j \le n+1,\ \forall v \in C^1(T).$$

In this situation, the corresponding basis functions remain unchanged and the corresponding finite element spaces are the same as the previous case.

Remark 4. Let e_{ij} $(1 \le i < j \le n+1)$ be the edge of T with a_i and a_j as its endpoints, and let $|e_{ij}|$ be its length and m_{ij} its midpoint. The n-dimensional Morley element given in [5] has the following degrees of freedom:

$$\mu v(m_{ij}) + \frac{1-\mu}{|e_{ij}|}\int_{e_{ij}} v,\ 1 \le i < j \le n+1,\ \ \frac{\partial v}{\partial \nu}(b_j),\ 1 \le j \le n+1,\ \forall v \in C^1(T), \quad (2)$$

where $\mu = 4 - 12/n$ and $n > 2$. This family of elements exclude the existing Morley element for $n = 2$. It is interesting to note that this element is identical to our new element for $n = 3$ but is quite different from our element for $n > 3$.

Lemma 1 *Given $v \in C^1(T)$, the degrees of freedom given in (1) uniquely determine the integrals of all first order derivatives*

$$\int_{F_j} \nabla v$$

on each $(n-1)$-dimensional subsimplex F_j of T.

Proof Given $1 \le j \le n+1$, denote the unit normal of F_j by ν, all $(n-2)$-dimensional subsimplexes of F_j by $S_1, S_2, \cdots, S_n$, and the unit out normal of S_i by $n^{(i)}$, viewed as the boundary of an $(n-1)$-simplex in $(n-1)$-dimensional space. Given any constant n-vector $\alpha \in R^n$, let $\tau = \alpha - (\alpha \cdot \nu)\nu$. Then $\tau \cdot \nu = 0$, namely τ is tangent to F. It follows that

$$\int_{F_j} \nabla p \cdot \alpha = (\alpha \cdot \nu)\int_{F_j} \frac{\partial p}{\partial \nu} + \sum_{i=1}^{n} \tau \cdot n^{(i)} \int_{S_i} p. \quad (3)$$

This gives an explicit expression of $\int_{F_j} \nabla p \cdot \alpha$ in terms of the degrees of freedom (1) for any $\alpha \in R^n$. The desired result then follows.

We now prove that the n-dimensional Morley element is well-defined.

Lemma 2 *For the Morley element of n-dimension, Φ_T is P_T-unisolvent.*

Proof Because the dimension of $P_2(T)$ and the number of degrees of freedom are all $\frac{1}{2}(n+1)(n+2)$, it is enough to show that if $p \in P_2(T)$ and

$$\int_{S_{ij}} p = 0,\ 1 \le i < j \le n+1,\ \ \int_{F_j} \frac{\partial p}{\partial \nu} = 0,\ 1 \le j \le n+1 \quad (4)$$

then $p \equiv 0$.

By Lemma 1 and its proof, we deduce that

$$\int_{F_j} \nabla p = 0, \quad 1 \le j \le n+1. \quad (5)$$

Now let $1 \le k, l \le n$. By Green's formula and (5) we have

$$\frac{\partial^2 p}{\partial x_k \partial x_l} = \frac{1}{|T|}\int_T \frac{\partial^2 p}{\partial x_k \partial x_l} = \frac{1}{|T|}\sum_{j=1}^{n+1}\int_{F_j} \frac{\partial p}{\partial x_k}\nu_l = 0.$$

That is, $p \in P_1(T)$. From (5), $\nabla p = 0$ and p is a constant on T. Hence $p \equiv 0$ by (4).

2.2 The nodal basis functions and the nodal value interpolant

For both theoretical and practical interests, we now give an explicit construction of the nodal basis functions for our new n-dimensional Morley element. Explicit nodal basis functions of course allow an explicit definition of the canonical nodal basis interpolant for the n-dimensional Morley element.

The nodal basis functions Let us first give the formulae for nodal basis functions.

Theorem 1 *The nodal basis functions associated with the degrees of freedom given by (1) for the n-dimensional Morley element are give by*

$$\begin{cases} q_i = \dfrac{1}{2\|\nabla\lambda_i\|}\lambda_i(n\lambda_i - 2), & 1 \le i \le n+1, \\ p_{ij} = 1 - (n-1)(\lambda_i + \lambda_j) + n(n-1)\lambda_i\lambda_j & \\ \qquad -(n-1)\nabla\lambda_i^\top\nabla\lambda_j \displaystyle\sum_{k=i,j} \dfrac{\lambda_k(n\lambda_k - 2)}{2\|\nabla\lambda_k\|^2}, & 1 \le i < j \le n+1, \end{cases} \tag{6}$$

where $\|\nabla\lambda_i\|$ is the Euclidean norm of $\nabla\lambda_i$.

Proof Let $1 \le i \le n+1$, $1 \le k < l \le n+1$. If $k = i$ or $l = i$ then $q_i|_{S_{kl}} = 0$. If $k \ne i$ and $l \ne i$ then

$$\frac{1}{|S_{kl}|}\int_{S_{kl}} q_i = \frac{(n-2)!}{2\|\nabla\lambda_i\|}\left(\frac{2n}{n!} - \frac{2}{(n-1)!}\right) = 0.$$

For $j \in \{1, 2, \cdots, n+1\}$, $-\nabla\lambda_j$ is the outer normal of F_j and the integral average of a linear polynomial over F_j equals to its value at point b_j. Since

$$\nabla q_i = \frac{1}{\|\nabla\lambda_i\|}(n\lambda_i - 1)\nabla\lambda_i, \quad \nabla q_i(b_j) = \begin{cases} -\dfrac{\nabla\lambda_i}{\|\nabla\lambda_i\|}, & j = i, \\ 0 & j \ne i, \end{cases}$$

we obtain that for $i \in \{1, 2, \cdots, n+1\}$

$$\begin{cases} \dfrac{1}{|S_{kl}|}\displaystyle\int_{S_{kl}} q_i = 0, \quad 1 \le k < l \le n+1, \\ \dfrac{1}{|F_k|}\displaystyle\int_{F_k} \dfrac{\partial q_i}{\partial\nu} = \delta_{ik}, \ 1 \le k \le n+1, \end{cases} \tag{7}$$

where δ_{ik} is the Kronecker delta.

Now let $1 \le i < j \le n+1$ and $1 \le k < l \le n+1$. If $k = i$ and $l = j$ then $p_{ij}|_{S_{kl}} = 1$. If $\{i, j\} \cap \{k, l\}$ has only one element, for example $k = i$ and $l \ne j$, then

$$\frac{1}{|S_{kl}|}\int_{S_{kl}} p_{ij} = \frac{1}{|S_{kl}|}\int_{S_{kl}} \Big(1 - (n-1)\lambda_j\Big) = (n-2)!\Big(\frac{1}{(n-2)!} - \frac{n-1}{(n-1)!}\Big) = 0$$

where we have used the first equality of (7). If $\{i,j\}\cap\{k,l\}$ is empty then

$$\frac{1}{|S_{kl}|}\int_{S_{kl}} p_{ij} = \frac{1}{|S_{kl}|}\int_{S_{kl}} \Big(1-(n-1)(\lambda_i+\lambda_j)+n(n-1)\lambda_i\lambda_j\Big)$$
$$=(n-2)!\Big(\frac{1}{(n-2)!}-\frac{2(n-1)}{(n-1)!}+\frac{n(n-1)}{n!}\Big)=0.$$

By virtue of the following equality

$$\nabla p_{ij}=(n-1)\Big(-\nabla\lambda_i-\nabla\lambda_j+n(\lambda_i\nabla\lambda_j+\lambda_j\nabla\lambda_i)-\nabla\lambda_i^\top\nabla\lambda_j\sum_{k=i,j}\frac{(n\lambda_k-1)\nabla\lambda_k}{\|\nabla\lambda_k\|^2}\Big)$$

we have

$$\nabla p_{ij}(b_k)=\begin{cases}-\nabla\lambda_j+\nabla\lambda_i^\top\nabla\lambda_j\dfrac{\nabla\lambda_i}{\|\nabla\lambda_i\|^2}, & k=i,\\ -\nabla\lambda_i+\nabla\lambda_i^\top\nabla\lambda_j\dfrac{\nabla\lambda_j}{\|\nabla\lambda_j\|^2}, & k=j,\\ 0 & k\neq i,j.\end{cases}$$

Therefore

$$\frac{\partial p_{ij}}{\partial\nu}(b_k)=0,\quad 1\le k\le n+1.$$

In summary, we have, for $1\le i<j\le n+1$

$$\begin{cases}\dfrac{1}{|S_{kl}|}\displaystyle\int_{S_{kl}} p_{ij}=\delta_{ik}\delta_{jl},\ 1\le k<l\le n+1,\\ \dfrac{1}{|F_k|}\displaystyle\int_{F_k}\frac{\partial p_{ij}}{\partial\nu}=0,\qquad 1\le k\le n+1.\end{cases} \tag{8}$$

From (7) and (8), p_{ij} and q_i are the nodal basis functions with respect to the degrees of freedom (1).

Theorem 1 can of course be used directly to give another proof of Lemma 2.

For practical interests, let us take a closer look at the nodal basis functions in three dimensional case. We note that $\nabla\lambda_i$ is a constant vector and can be represented by the components of vertices. Set

$$c_1=\begin{pmatrix}(x_{22}-x_{23})(x_{34}-x_{33})-(x_{23}-x_{24})(x_{33}-x_{32})\\(x_{13}-x_{12})(x_{34}-x_{33})-(x_{14}-x_{13})(x_{33}-x_{32})\\(x_{13}-x_{12})(x_{23}-x_{24})-(x_{14}-x_{13})(x_{22}-x_{23})\end{pmatrix},$$

$$c_2=\begin{pmatrix}(x_{23}-x_{21})(x_{34}-x_{33})-(x_{24}-x_{23})(x_{33}-x_{31})\\(x_{11}-x_{13})(x_{34}-x_{33})-(x_{13}-x_{14})(x_{33}-x_{31})\\(x_{11}-x_{13})(x_{23}-x_{24})-(x_{13}-x_{14})(x_{21}-x_{23})\end{pmatrix},$$

$$c_3 = \begin{pmatrix} (x_{21}-x_{22})(x_{34}-x_{32})-(x_{22}-x_{24})(x_{32}-x_{31}) \\ (x_{12}-x_{11})(x_{34}-x_{32})-(x_{14}-x_{12})(x_{32}-x_{31}) \\ (x_{12}-x_{11})(x_{22}-x_{24})-(x_{14}-x_{12})(x_{21}-x_{22}) \end{pmatrix},$$

$$c_4 = \begin{pmatrix} (x_{22}-x_{21})(x_{33}-x_{32})-(x_{23}-x_{22})(x_{32}-x_{31}) \\ (x_{11}-x_{12})(x_{33}-x_{32})-(x_{12}-x_{13})(x_{32}-x_{31}) \\ (x_{11}-x_{12})(x_{22}-x_{23})-(x_{12}-x_{13})(x_{21}-x_{22}) \end{pmatrix}.$$

Then, for the 3-dimensional Morley element, its nodal basis function can be written as

$$\begin{cases} q_i = \dfrac{3|T|}{\|c_i\|}\lambda_i(3\lambda_i-2), \quad 1 \le i \le 4, \\ p_{ij} = 1-2(\lambda_i+\lambda_j)+6\lambda_i\lambda_j - c_i^\top c_j \displaystyle\sum_{k=i,j} \dfrac{\lambda_k(3\lambda_k-2)}{\|c_k\|^2}, \ \ 1 \le i < j \le 4. \end{cases} \tag{9}$$

The nodal value interpolant With the nodal basis functions given above, the corresponding interpolation operator Π_T can then be given by

$$\Pi_T v = \sum_{1\le i<j\le n+1} \frac{p_{ij}}{|S_{ij}|}\int_{S_{ij}} v + \sum_{j=1}^{n+1} \frac{q_j}{|F_j|}\int_{F_j} \frac{\partial v}{\partial \nu}, \ \forall v \in H^2(T). \tag{10}$$

By construction, we have

$$\Pi_T p = p, \quad \forall p \in P_2(T). \tag{11}$$

Using (11) and the interpolation theory [2], we obtain the following lemma.

Lemma 3 *For the n-dimensional Morley element, there exists a constant C independent of h such that*

$$|v - \Pi_T v|_{m,T} \le C h^{3-m} |v|_{3,T}, \quad 0 \le m \le 3, \ \forall v \in H^3(T), \ T \in \mathcal{T}_h. \tag{12}$$

Define Π_h by $(\Pi_h v)|_T = \Pi_T(v|_T)$, $\forall T \in \mathcal{T}_h$, where v is appropriately smooth. By (7), (8) and (10), we have, for $v \in H^2(T)$,

$$\begin{cases} \dfrac{1}{|S_{ij}|}\displaystyle\int_{S_{ij}} \Pi_T v = \dfrac{1}{|S_{ij}|}\displaystyle\int_{S_{ij}} v, \quad 1 \le i < j \le n+1, \\ \dfrac{1}{|F_j|}\displaystyle\int_{F_j} \dfrac{\partial \Pi_T v}{\partial \nu} = \dfrac{1}{|F_j|}\displaystyle\int_{F_j} \dfrac{\partial v}{\partial \nu}, 1 \le j \le n+1. \end{cases} \tag{13}$$

The n-dimensional Morley finite element space Let h_T be the diameter of the smallest ball containing T and ρ_T be the diameter of the largest ball contained in T. Let $\{\mathcal{T}_h\}$ be a family of triangulations of Ω, consisting of n-simplexes, with mesh size $h \to 0$. Throughout the paper, we assume that $\{\mathcal{T}_h\}$ satisfies: $h_T \le h$, $\forall T \in \mathcal{T}_h$, and there exists a positive constant η independent of h, such that $\eta h \le \rho_T$, $\forall T \in \mathcal{T}_h$.

For the n-dimensional Morley element, the corresponding finite element spaces V_h and V_{h0} are defined as follows. V_h consists of all piecewise quadratic functions on $\mathcal{T}_h$ such that, their integral average over each $(n-2)$-dimensional subsimplex of elements in $\mathcal{T}_h$ are continuous, and their normal derivatives are continuous at the barycentric point of each $(n-1)$-dimensional subsimplex of elements in $\mathcal{T}_h$, and V_{h0} consists of functions in V_h whose degrees of freedom (1) vanish on $\partial\Omega$.

2.3 Some properties

For $v_h \in V_h$ and $T \in \mathcal{T}_h$, denote by v_h^T the continuous extension of v_h from the interior of T to T. Given any $(n-1)$-dimensional subsimplex F, let us define the jumps of v_h and ∇v_h across F as follows:

$$[v_h] = v_h^T - v_h^{T'} \text{ and } [\nabla v_h] = \nabla v_h^T - \nabla v_h^{T'}$$

if $F = T \cap T'$ for some $T, T' \in \mathcal{T}_h$ and

$$[v_h] = v_h^T \text{ and } [\nabla v_h] = \nabla v_h^T$$

if $F = T \cap \partial\Omega$.

The first property we will state now is a direct consequence of Lemma 1.

Lemma 4 *If F is a common $(n-1)$-dimensional subsimplex of $T, T' \in \mathcal{T}_h$, then*

$$\int_F [\nabla v_h] = 0, \quad \forall v_h \in V_h. \tag{14}$$

If an $(n-1)$-dimensional subsimplex F of $T \in \mathcal{T}_h$ is on $\partial\Omega$ then

$$\int_F [\nabla v_h] = 0, \quad \forall v_h \in V_{h0}. \tag{15}$$

Lemma 5 *There exists a constant C independent of h such that*

$$\big\|[v_h]\big\|_{0,F} + h\big\|[\nabla v_h]\big\|_{0,F} \le Ch^{3/2}\Big(|v_h|_{2,T} + |v_h|_{2,T'}\Big), \quad \forall v_h \in V_h \tag{16}$$

if $F = T \cap T'$ is a common $(n-1)$-dimensional subsimplex of some $T, T' \in \mathcal{T}_h$, and

$$\big\|[v_h]\big\|_{0,F} + h\big\|[\nabla v_h]\big\|_{0,F} \le Ch^{3/2}|v_h|_{2,T}, \quad \forall v_h \in V_{h0} \tag{17}$$

if $F = T \cap \partial\Omega$.

Proof Let $v_h \in V_h$ and $F = T \cap T'$. From (14) we know that $[\nabla v_h]$ vanishes at a point on F. Then

$$\max_{x \in F} \big\|[\nabla v_h](x)\big\| \le h \max_{x \in F} \sum_{i,j=1}^{n} \Big|\Big[\frac{\partial v_h}{\partial x_i \partial x_j}\Big](x)\Big|. \tag{18}$$

By a standard scaling argument (or inverse inequality), we obtain

$$\big\|[\nabla v_h]\big\|_{0,F} \le Ch^{1/2}\Big(|v_h|_{2,T} + |v_h|_{2,T'}\Big). \tag{19}$$

From the definition of V_h, $[v_h]$ vanishes at some point on F. Then

$$\big\|[v_h]\big\|_{0,F} \le h^{(n-1)/2} \max_{x\in F} \big|[v_h](x)\big| \le h^{(n+1)/2} \max_{x\in F} \big\|[\nabla v_h](x)\big\| \le Ch\big\|[\nabla v_h]\big\|_{0,F}. \tag{20}$$

Inequality (19) leads to

$$\big\|[v_h]\big\|_{0,F} \le Ch^{3/2}\Big(|v_h|_{2,T} + |v_h|_{2,T'}\Big). \tag{21}$$

Inequality (16) follows from (19) and (21).

Let $v_h \in V_{h0}$ and $F = T \cap \partial\Omega$. Then $[v_h] = v_h|_F$ and $[\nabla v_h] = \nabla v_h|_F$. From the definition of V_{h0} and (15), $[v_h]$ and $[\nabla v_h]$ vanish at some points on F respectively. Then inequalities (18) and (20) can be proved similarly in this case. Thus inequality (17) is true.

3 The convergence analysis for the biharmonic equations

For $f \in L^2(\Omega)$, we consider the following boundary value problem of the biharmonic equation

$$\begin{cases} \Delta^2 u = f, & \text{in } \Omega, \\ u|_{\partial\Omega} = \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0 \end{cases} \tag{22}$$

where Δ is the standard Laplacian operator. Define

$$a(v,w) = \int_\Omega \sum_{i,j=1}^n \frac{\partial^2 v}{\partial x_i \partial x_j}\frac{\partial^2 w}{\partial x_i \partial x_j}, \quad \forall v, w \in H^2(\Omega). \tag{23}$$

The weak form of problem (22) is: find $u \in H_0^2(\Omega)$ such that

$$a(u,v) = (f,v), \qquad \forall v \in H_0^2(\Omega). \tag{24}$$

The 2-dimensional Morley element is a convergent element for the fourth order elliptic equations (see [2,3,6,7]), while it is divergent in general for the second order equations (see [8]). In this section, we discuss some convergence properties of the n-dimensional Morley element for problem (22). The main idea of the analysis follows from Shi [6].

We introduce the following mesh dependent norm $\|\cdot\|_{m,h}$ and semi-norm $|\cdot|_{m,h}$:

$$\|v\|_{m,h} = \Big(\sum_{T\in\mathcal{T}_h} \|v\|_{m,T}^2\Big)^{1/2}, \quad |v|_{m,h} = \Big(\sum_{T\in\mathcal{T}_h} |v|_{m,T}^2\Big)^{1/2}$$

for all function $v \in L^2(\Omega)$ that $v|_T \in H^m(T),\ \forall T \in \mathcal{T}_h$.

Lemma 6 *For any $v_h \in V_{h0}$ there exist functions $w_{hk} \in H_0^1(\Omega)$, $0 \le k \le n$, such that $w_{hk}|_T \in C^\infty(T), \forall T \in \mathcal{T}_h$, and*

$$|v_h - w_{h0}|_{m,h} \le Ch^{2-m}|v_h|_{2,h}, \quad 0 \le m \le 2, \tag{25}$$

$$\Big|\frac{\partial v_h}{\partial x_k} - w_{hk}\Big|_{m,h} \le Ch^{1-m}|v_h|_{2,h}, \quad 0 \le m \le 1,\ 1 \le k \le n \tag{26}$$

where C is a constant independent of h.

Proof Let $v_h \in V_{h0}$, and let $P_T^1 : L^2(T) \to P_1(T)$ be the L^2-orthogonal projection. Define $P_h^1 : L^2(\Omega) \to L^2(\Omega)$ as follows: for any $v \in L^2(\Omega)$, $P_h^1 v|_T = P_T^1 v$, $\forall T \in \mathcal{T}_h$. Set

$$\phi_{h0} = P_h^1 v_h,\ \phi_{hk} = P_h^1 \frac{\partial v_h}{\partial x_k},\ 1 \le k \le n.$$

By a standard error analysis, we have

$$\begin{cases} |v_h - \phi_{h0}|_{m,h} \le Ch^{2-m}|v_h|_{2,h}, \quad 0 \le m \le 2, \\ \Big|\dfrac{\partial v_h}{\partial x_k} - \phi_{hk}\Big|_{m,h} \le Ch^{1-m}|v_h|_{2,h}, 0 \le m \le 1,\ 1 \le k \le n. \end{cases} \tag{27}$$

Given a set $B \subset R^n$, let $\mathcal{T}_h(B) = \{ T \in \mathcal{T}_h \mid B \cap T \neq \emptyset \}$ and $N_h(B)$ the number of the elements in $\mathcal{T}_h(B)$.

For $k \in \{0, 1, \cdots, n\}$, we define $w_{hk} \in H_0^1(\Omega)$ as follows: for any $T \in \mathcal{T}_h$, $w_{hk}|_T \in P_1(T)$ and for $i \in \{1, 2, \cdots, n+1\}$

$$w_{hk}(a_i) = \frac{1}{N_h(a_i)} \sum_{T' \in \mathcal{T}_h(a_i)} \phi_{hk}^{T'}(a_i)$$

when vertex a_i of T is in Ω. Obviously, w_{hk} is well-defined. To prove the lemma, we only need to show that

$$\begin{cases} |v_h - w_{h0}|_{m,h} \le Ch^{2-m}|v|_{2,h}, \quad 0 \le m \le 2, \\ \Big|\dfrac{\partial v_h}{\partial x_k} - w_{hk}\Big|_{m,h} \le Ch^{1-m}|v_h|_{2,h}, 0 \le m \le 1,\ 1 \le k \le n. \end{cases} \tag{28}$$

Let $T \in \mathcal{T}_h$, by a standard scaling argument, we have

$$|p|_{m,T}^2 \le Ch^{n-2m} \sum_{i=1}^{n+1} |p(a_i)|^2, \quad 0 \le m \le 2,\ \forall p \in P_1(T). \tag{29}$$

If vertex a_i of T is in Ω then by the definition of w_{hk},

$$\begin{aligned}(\phi_{hk}^T - w_{hk})(a_i) &= \phi_{hk}^T(a_i) - \frac{1}{N_h(a_i)} \sum_{T' \in \mathcal{T}_h(a_i)} \phi_{hk}^{T'}(a_i) \\ &= \frac{1}{N_h(a_i)} \sum_{T' \in \mathcal{T}_h(a_i)} \Big(\phi_{hk}^T(a_i) - \phi_{hk}^{T'}(a_i)\Big).\end{aligned}$$

For $T' \in \mathcal{T}_h(a_i)$ there exist $T_1, \cdots, T_J \in \mathcal{T}_h(a_i)$ such that $T_1 = T$, $T_J = T'$ and $\tilde{F}_j = T_j \cap T_{j+1}$ is a common $(n-1)$-dimensional subsimplex of T_j and T_{j+1} and $a_i \in \tilde{F}_j$, $1 \le j < J$. By standard inverse inequalities, we have

$$\Big|\phi_{h0}^T(a_i) - \phi_{h0}^{T'}(a_i)\Big|^2 = \Big|\sum_{j=1}^{J-1} \big(\phi_{h0}^{T_j}(a_i) - \phi_{h0}^{T_{j+1}}(a_i)\big)\Big|^2$$

$$\leq C \sum_{j=1}^{J-1} \left| \phi_{h0}^{T_j}(a_i) - \phi_{h0}^{T_{j+1}}(a_i) \right|^2 \leq Ch^{1-n} \sum_{j=1}^{J-1} \left| \phi_{h0}^{T_j} - \phi_{h0}^{T_{j+1}} \right|_{0,\tilde{F}_j}^2$$

$$\leq Ch^{1-n} \sum_{j=1}^{J-1} \Big(\left| v_h^{T_j} - \phi_{h0}^{T_j} \right|_{0,\tilde{F}_j}^2 + \left| v_h^{T_{j+1}} - \phi_{h0}^{T_{j+1}} \right|_{0,\tilde{F}_j}^2 + \left| [v_h] \right|_{0,\tilde{F}_j}^2 \Big).$$

Similarly,

$$\left| \phi_{hk}^{T}(a_i) - \phi_{hk}^{T'}(a_i) \right|^2$$

$$\leq Ch^{1-n} \sum_{j=1}^{J-1} \Big(\Big| \frac{\partial v_h^{T_j}}{\partial x_k} - \phi_{hk}^{T_j} \Big|_{0,\tilde{F}_j}^2 + \Big| \frac{\partial v_h^{T_{j+1}}}{\partial x_k} - \phi_{hk}^{T_{j+1}} \Big|_{0,\tilde{F}_j}^2 + \left| [\nabla v_h] \right|_{0,\tilde{F}_j}^2 \Big)$$

when $1 \leq k \leq n$. By a standard analysis, we obtain

$$\left| \phi_{h0}^{T}(a_i) - \phi_{h0}^{T'}(a_i) \right|^2 \leq Ch^{1-n} \Big(h^3 \sum_{j=1}^{J} |v_h|_{2,T_j}^2 + \sum_{j=1}^{J-1} \left| [v_h] \right|_{0,\tilde{F}_j}^2 \Big),$$

$$\left| \phi_{hk}^{T}(a_i) - \phi_{hk}^{T'}(a_i) \right|^2 \leq Ch^{1-n} \Big(h \sum_{j=1}^{J} |v_h|_{2,T_j}^2 + \sum_{j=1}^{J-1} \left| [\nabla v_h] \right|_{0,\tilde{F}_j}^2 \Big),\ 1 \leq k \leq n.$$

From Lemma 5 we have

$$\left| \phi_{h0}^{T}(a_i) - \phi_{h0}^{T'}(a_i) \right|^2 \leq Ch^{4-n} \sum_{j=1}^{J} |v|_{2,T_j}^2,$$

$$\left| \phi_{hk}^{T}(a_i) - \phi_{hk}^{T'}(a_i) \right|^2 \leq Ch^{2-n} \sum_{j=1}^{J} |v|_{2,T_j}^2,\ 1 \leq k \leq n.$$

Since $N_h(T)$ is bounded, we get

$$\begin{cases} |(\phi_{h0}^{T} - w_{h0})(a_i)|^2 \leq Ch^{4-n} \displaystyle\sum_{T' \in \mathcal{T}_h(T)} |v|_{2,T'}^2, \\ |(\phi_{hk}^{T} - w_{hk})(a_i)|^2 \leq Ch^{2-n} \displaystyle\sum_{T' \in \mathcal{T}_h(T)} |v|_{2,T'}^2,\ 1 \leq k \leq n. \end{cases} \tag{30}$$

If vertex a_i of T is on $\partial\Omega$, there exists $T' \in \mathcal{T}_h(a_i)$ with an $(n-1)$-dimensional sub-simplex F of T' belonging to $\partial\Omega$ and $a_i \in F$. By the definitions of w_{hk} and ϕ_{hk},

$$|(\phi_{h0}^{T} - w_{h0})(a_i)| \leq |\phi_{h0}^{T}(a_i) - \phi_{h0}^{T'}(a_i)| + |v_h^{T'}(a_i) - \phi_{h0}^{T'}(a_i)| + |v_h^{T'}(a_i)|.$$

and for $1 \le k \le n$

$$|(\phi_{hk}^T - w_{hk})(a_i)| \le |\phi_{hk}^T(a_i) - \phi_{hk}^{T'}(a_i)| + \Big|\frac{\partial v_h^{T'}}{\partial x_k}(a_i) - \phi_{hk}^{T'}(a_i)\Big| + \Big|\frac{\partial v_h^{T'}}{\partial x_k}(a_i)\Big|.$$

By scaling argument and Lemma 5, we have

$$|v_h^{T'}(a_i)|^2 \le Ch^{4-n}|v_h|_{2,T'}^2,$$

$$\Big|\frac{\partial v_h^{T'}}{\partial x_k}(a_i)\Big|^2 \le Ch^{2-n}|v_h|_{2,T'}^2,\ 1 \le k \le n.$$

Using a routine analysis, we have

$$|v_h^{T'}(a_i) - \phi_{h0}^{T'}(a_i)|^2 \le Ch^{4-n}|v_h|_{2,T'}^2,$$

$$\Big|\frac{\partial v_h^{T'}}{\partial x_k}(a_i) - \phi_{hk}^{T'}(a_i)\Big|^2 \le Ch^{2-n}|v_h|_{2,T'}^2,\ 1 \le k \le n.$$

By a similar analysis for $|\phi_{hk}^T(a_i) - \phi_{hk}^{T'}(a_i)|$, $0 \le k \le n$, we conclude that (30) is also true in this case.

Combining (29) and (30), we have

$$h^{2m}|\phi_{h0} - w_{h0}|_{m,T}^2 \le Ch^4 \sum_{T' \in \mathcal{T}_h(T)} |v_h|_{2,T'}^2,$$

$$h^{2m}|\phi_{hk} - w_{hk}|_{m,T}^2 \le Ch^2 \sum_{T' \in \mathcal{T}_h(T)} |v_h|_{2,T'}^2,\ 1 \le k \le n.$$

Summing the above inequalities over all $T \in \mathcal{T}_h$, we obtain that

$$h^{2m}|\phi_{h0} - w_{h0}|_{m,h}^2 \le Ch^4 \sum_{T \in \mathcal{T}_h} \sum_{T' \in \mathcal{T}_h(T)} |v_h|_{2,T'}^2,$$

$$h^{2m}|\phi_{hk} - w_{hk}|_{m,h}^2 \le Ch^2 \sum_{T \in \mathcal{T}_h} \sum_{T' \in \mathcal{T}_h(T)} |v_h|_{2,T'}^2,\ 1 \le k \le n.$$

Consequently

$$h^{2m}|\phi_{h0} - w_{h0}|_{m,h}^2 \le Ch^4|v_h|_{2,h}^2, \tag{31}$$

$$h^{2m}|\phi_{hk} - w_{hk}|_{m,h}^2 \le Ch^2|v_h|_{2,h}^2, \quad 1 \le k \le n. \tag{32}$$

Then (28) follows from (31), (32) and (27).

For $v, w \in H^2(\Omega) + V_h$, we define

$$a_h(v, w) = \sum_{T \in \mathcal{T}_h} \int_T \sum_{i,j=1}^n \frac{\partial^2 v}{\partial x_i \partial x_j} \frac{\partial^2 w}{\partial x_i \partial x_j}. \tag{33}$$

The finite element method for problem (24) is: find $u_h \in V_{h0}$ such that

$$a_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_{h0}. \tag{34}$$

Lemma 7 *There exists a constant C independent of h such that for $v \in H^3(\Omega) \cap H_0^2(\Omega)$ with $\Delta^2 v \in L^2(\Omega)$,*

$$|a_h(v, v_h) - (\Delta^2 v, v_h)| \le Ch(|v|_{3,\Omega} + h\|\Delta^2 v\|_{0,\Omega})|v_h|_{2,h}, \quad \forall v_h \in V_{h0}. \tag{35}$$

Proof For $v \in H^3(\Omega) \cap H_0^2(\Omega)$ with $\Delta^2 v \in L^2(\Omega)$ and $v_h \in V_{h0}$, let $w_{h0} \in H_0^1(\Omega)$ be as in (25). We write

$$a_h(v, v_h) - (\Delta^2 v, v_h) = \Big(a_h(v, v_h) - (\Delta^2 v, w_{h0})\Big) + (\Delta^2 v, w_{h0} - v_h). \tag{36}$$

By (25) and the Schwarz inequality we obtain immediately that

$$|(\Delta^2 v, w_{h0} - v_h)| \le Ch^2 \|\Delta^2 v\|_{0,\Omega} |v_h|_{2,h}. \tag{37}$$

For the first term on the right of (36), an integration by parts gives

$$\begin{aligned} a_h(v, v_h) - (\Delta^2 v, w_{h0}) = &\sum_{i,j=1}^{n} \sum_{T \in \mathcal{T}_h} \int_T \Big(\frac{\partial^2 v}{\partial x_i \partial x_j} \frac{\partial^2 v_h}{\partial x_i \partial x_j} + \frac{\partial^3 v}{\partial x_i \partial x_j^2} \frac{\partial v_h}{\partial x_i} \Big) \\ &+ \sum_{i,j=1}^{n} \sum_{T \in \mathcal{T}_h} \int_T \frac{\partial^3 v}{\partial x_i \partial x_j^2} \frac{\partial (w_{h0} - v_h)}{\partial x_i}. \end{aligned} \tag{38}$$

Now let $i, j \in \{1, 2, \cdots, n\}$. By (25) and the Schwarz inequality we have

$$\left| \sum_{T \in \mathcal{T}_h} \int_T \frac{\partial^3 v}{\partial x_i \partial x_j^2} \frac{\partial (w_{h0} - v_h)}{\partial x_i} \right| \le Ch|v|_{3,\Omega} |v_h|_{2,h}. \tag{39}$$

For an $(n-1)$-subsimplex F_k of $T \in \mathcal{T}_h$, let $P_{F_k}^0 : L^2(F_k) \to P_0(F_k)$ be the L^2-orthogonal projection. By Green's formula and Lemma 4, we have

$$\begin{aligned} \sum_{T \in \mathcal{T}_h} \int_T \Big(\frac{\partial^2 v}{\partial x_i \partial x_j} \frac{\partial^2 v_h}{\partial x_i \partial x_j} + \frac{\partial^3 v}{\partial x_i \partial x_j^2} \frac{\partial v_h}{\partial x_i} \Big) &= \sum_{T \in \mathcal{T}_h} \sum_{k=1}^{n+1} \int_{F_k} \frac{\partial^2 v}{\partial x_i \partial x_j} \frac{\partial v_h}{\partial x_i} \nu_j \\ &= \sum_{T \in \mathcal{T}_h} \sum_{k=1}^{n+1} \int_{F_k} \Big(\frac{\partial^2 v}{\partial x_i \partial x_j} - P_{F_k}^0 \frac{\partial^2 v}{\partial x_i \partial x_j} \Big) \Big(\frac{\partial v_h}{\partial x_i} - P_{F_k}^0 \frac{\partial v_h}{\partial x_i} \Big) \nu_j, \end{aligned}$$

which implies that

$$\left| \sum_{T \in \mathcal{T}_h} \int_T \Big(\frac{\partial^2 v}{\partial x_i \partial x_j} \frac{\partial^2 v_h}{\partial x_i \partial x_j} + \frac{\partial^3 v}{\partial x_i \partial x_j^2} \frac{\partial v_h}{\partial x_i} \Big) \right| \le Ch|v|_{3,\Omega} |v_h|_{2,h}. \tag{40}$$

Equality (38) together with (39) and (40) leads to

$$\Big| a_h(v, v_h) - (\Delta^2 v, w_{h0}) \Big| \le Ch|v|_{3,\Omega} |v_h|_{2,h}. \tag{41}$$

Inequality (35) follows from (36), (37) and (41).

Lemma 8 *There exists a constant C independent of h such that, for any $v_h \in V_{h0}$,*

$$|v_h|_{2,h} \leq \|v_h\|_{2,h} \leq C|v_h|_{2,h}. \tag{42}$$

Proof For $v_h \in V_{h0}$, let $w_{hk} \in H_0^1(\Omega)$, $0 \leq k \leq n$, such that inequalities (25) and (26) hold. Then from (25) and (26)

$$\|v_h\|_{0,h} \leq \|v_h - w_{h0}\|_{0,h} + \|w_{h0}\|_{0,\Omega} \leq C(|v_h|_{2,h} + |w_{h0}|_{1,\Omega}) \leq C(|v_h|_{2,h} + |v_h|_{1,h}),$$

$$|v_h|_{1,h} \leq \sum_{k=1}^{n} \Big(\Big|\frac{\partial v_h}{\partial x_k} - w_{hk}\Big|_{0,h} + \|w_{hk}\|_{0,\Omega}\Big) \leq C\Big(|v_h|_{2,h} + \sum_{k=1}^{n} |w_{hk}|_{1,\Omega}\Big) \leq C|v_h|_{2,h}.$$

The above inequalities lead to the second inequality of (42).

Theorem 2 *Let u and u_h be the solutions of problem (24) and (34) respectively. Then there exists a constant C independent of h such that*

$$\|u - u_h\|_{2,h} \leq Ch(|u|_{3,\Omega} + h\|f\|_{0,\Omega}) \tag{43}$$

when $u \in H^3(\Omega)$.

Proof The well-known Strang's Lemma (see [7] or [2]) says that

$$|u - u_h|_{2,h} \leq C \left(\inf_{w_h \in V_{h0}} |u - w_h|_{2,h} + \sup_{w_h \in V_{h0}\, w_h \neq 0} \frac{|\, a_h(u, w_h) - (f, w_h)|}{|w_h|_{2,h}} \right). \tag{44}$$

By (42), we may replace the semi-norm $|\cdot|_{2,h}$ above by the full norm $\|\cdot\|_{2,h}$. The desired estimate (43) then follows from Lemma 3 and Lemma 7.

4 Concluding remarks

The two dimensional nonconforming Morley element is a very simple but peculiar element for biharmonic equations. In this paper, this element is extended to the general n-dimensional case in a canonical fashion. The new class of nonconforming elements constructed in this paper for fourth order partial differential equations is hoped to shed some new insight to the finite element theory on nonconforming elements. In addition to its theoretical interest, as pointed out in [9], this type of element is potentially useful in practice such as in computational material sciences.

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