微分拓扑, 2025 年春季

作业 2

上交时间及方式：2025.05.09

REQUIRED:

1. Let X and Z be transversal submanifolds of Y. Show that if $y \in X \cap Z$, then $T_y(X \cap Z) = T_y(X) \cap T_y(Z)$, that is, the tangent space to the intersection is the intersection of the tangent spaces.

2. If $f : M \to Y$ and $g : N \to Y$ are smooth morphisms of manifolds, the fiber product $M \times_Y N$ is defined by
$$M \times_Y N = \{(m, n) \mid m \in M, n \in N, f(m) = g(n)\}$$
We say that f and g are transverse if for all pairs $(m, n) \in M \times_Y N$ we have that $Df_m(T_m M) + Dg_n(T_n N) = T_{f(m)}Y$. Show that if $g : N \to Y$ is an embedding of a submanifold, that f is transverse to g in this sense if and only if f is transverse to $g(N)$. Show that if f and g are transverse, the fiber product $M \times_Y N$ is a manifold.

   (Hint: Consider the map $f \times g : M \times N \to Y \times Y$.)

3. If $M \subset \mathbb{R}^n$ is a submanifold, define the tangent bundle to M to be the set
$$TM = \{(m, v) \mid m \in M, v \in T_m M\} \subset M \times R^n$$
Show that TM is a smooth manifold. Show that if $f : M \to N$ is a smooth morphism, that the natural map $Df : TM \to TN$ defined by $Df(m, v) = (f(m), Df_m(v))$ is a smooth morphism. Conclude that if M is diffeomorphic to N, then TM is diffeomorphic to TN. (So that TM does not depend on how M was embedded into $\mathbb{R}^n$.)

4. Show that diffeomorphism is a stable property for maps of compact manifolds. That is, if $f_0 : X \to Y$ is a diffeomorphism between compact smooth manifolds, and $f_t : X \to Y$ is a homotopy of f_0, then, for some $\varepsilon > 0$, each f_t with $t < \varepsilon$ is also a diffeomorphism. Here we suppose X and Y are connected.

5. Suppose Z is a submanifold of X with $\dim Z < \dim X$. Show that Z has measure zero in X WITHOUT using Sard.

6. Let X be a compact manifold. Show that there exist Morse functions on X which take distinct values at distinct critical points.

   (Hint: Let $x_1, \cdots, x_N$ be the list of critical points of f. Let ρ_i be a smooth function which is one on a small neighborhood of x_i and zero outside a slightly larger neighborhood. Choose numbers $a_1, \cdots, a_N$ such that
$$f(x_i) + a_i \neq f(x_j) + a_j \quad \text{if} \quad i \neq j$$
Show that for a_i small enough, the function $f + \sum a_i \rho_i$ is Morse and has the same critical points as f and is even arbitrarily close to f.)

7. A *vector field* $\vec{v}$ on a manifold X in $\mathbb{R}^N$ is a smooth map $\vec{v} : X \to \mathbb{R}^N$ such that $\vec{v}(x)$ is always tangent to X at x. Show that a vector field $\vec{v}$ is a cross section of $T(X)$ (that is, a smooth map $\vec{v} : X \to T(X)$ such that $p \circ \vec{v} = \mathbb{1}_X$), and vice versa. Here p is the projection map $p : T(X) \to X$ defined by $p(x, v) = x$.

8. (*The Whitney Immersion Theorem*) Show that every k-dimensional manifold X may be immersed into $\mathbb{R}^{2k}$.

9. (*The Smooth Urysohn Theorem*) If A and B are disjoint, smooth, closed subsets of a manifold X, prove that there is a smooth function ϕ on X such that $0 \leq \phi \leq 1$ with $\phi = 0$ on A and $\phi = 1$ on B.

   (Hint: Partition of Unity.)

SUGGESTED:

1. * If f and g are two smooth maps from a smooth manifold M to S^n, show that if for all $m \in M$ one has that $\|f(m) - g(m)\| < 2$ then f is smoothly homotopic to g.

2. * Exhibit a smooth map $f : \mathbb{R} \to \mathbb{R}$ whose set of critical values is dense.

   (Hint: Write the rationals in a sequence $r_0, r_1, \cdots$. Now construct a smooth function on $[i, i+1]$ that is zero near the endpoints and that has r_i as a critical value.)

3. * Show that the sphere S^k is simply connected if $k > 1$ using Sard.

   (Hint: If $f : S^1 \to S^k$ and $k > 1$, Sard gives you a point $p \notin f(S^1)$. Now use stereographic projection.)

4. * Show that $T(X \times Y)$ is diffeomorphic to $T(X) \times T(Y)$, and the projection $p : T(X) \to X, p(x, v) = x$ is a submersion.

5. * Let $S(X)$ be the set of points $(x, v) \in T(X)$ with $|v| = 1$. Show that $S(X)$ is a smooth submanifold of $T(X)$ of dimension $2k - 1$. $S(X)$ is called the *unit tangent bundle* or *sphere tangent bundle* of X.

   (Hint: Consider the map $(x, v) \mapsto |v|$.)