微分拓扑, 2025 年春季

作业 1

上交时间及方式：2025.03.14

1. Let $X \subset \mathbb{R}^N$, $Y \subset \mathbb{R}^M$ and $Z \subset \mathbb{R}^L$ be arbitrary subsets in certain Euclidean spaces, and let $f : X \to Y$ and $g : Y \to Z$ be smooth maps. Show that the composite $g \circ f : X \to Z$ is smooth, and if f and g are diffeomorphisms, so is $g \circ f$.

2. The *graph* of a map $f : X \to Y$ is the subset of $X \times Y$ defined by
$$\text{graph}(f) = \{(x, f(x)) : x \in X\}.$$
Define $F : X \to \text{graph}(f)$ by $F(x) = (x, F(x))$. Show that if f is smooth, then F is a diffeomorphism.; thus graph(f) is a manifold if X is.

3. (a) An extremely useful function $f : \mathbb{R}^1 \to \mathbb{R}^1$ is
$$f(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \leq 0 \end{cases}$$
Prove that f is smooth.
   (b) Show that $g(x) = f(x-a)f(b-x)$ is a smooth function, positive on (a, b) and zero elsewhere. (Here $a < b$.) Then
$$h(x) = \frac{\int_{-\infty}^{x} g(t)}{\int_{-\infty}^{\infty} g(t)}$$
is a smooth function satisfying $h(x) = 0$ for $x < a$, $h(x) = 1$ for $x > b$, and $0 < h(x) < 1$ for $x \in (a, b)$.
   (c) Now construct a smooth function on $\mathbb{R}^k$ that equals 1 on the ball of radius a zero outside the ball of radio b, and is strictly between 0 and 1 at intermediate points. (Here $0 < a < b$.)

4. If U is an open subset of the manifold X, check that
$$T_x(U) = T_x(X) \quad \text{for} \quad x \in U.$$

5. (a) Let $f : X \to X \times X$ be the mapping $f(x) = (x, x)$. Check that $df_x(v) = (v, v)$.
   (b) Let $\Delta = \{(x, x) : x \in X\}$ be the diagonal of $X \times X$, show that its tangent space $T_{(x,x)}(\Delta)$ is the diagonal of $T_x(X) \times T_x(X)$.

6. (a) Suppose that $f : X \to Y$ is a smooth map, and let $F : X \to Y$ be $F(x) = (x, f(x))$. Show that $dF_x(v) = (v, df_x(v))$.
   (b) Prove that the tangent space to graph(f) at the point $(x, f(x))$ is the graph of $df_x : T_x(X) \to T_{f(x)}(Y)$.

7. (a) Let $x_1, \cdots, x_N$ be the standard coordinate functions on $\mathbb{R}^N$, and let X be a k-dimensional submanifold of $\mathbb{R}^N$. Prove that every point $x \in X$ has a neighborhood on which the restrictions of some k-coordinate functions $x_{i_1}, \cdots, x_{i_k}$ form a local coordinate system.
   (b) For simplicity, assume that $x_1, \cdots, x_k$ form a local coordinate system on a neighborhood V of x in X. Show that there are smooth functions $g_{k+1}, \cdots, g_N$ on an open set U in $\mathbb{R}^K$ such that V may be taken to be the set
$$\Big\{\big(a_1, \cdots, a_k, g_{k+1}(a), \cdots, g_N(a)\big) \in \mathbb{R}^N : a = (a_1 \cdots, a_k) \in U\Big\}$$

8. If X is compact and Y is connected, show that every submersion $f : X \to Y$ is surjective. Show that there exist no submersions of compact manifolds into Euclidean spaces.

9. Verify that the tangent space to $O(n)$ at the identity matrix I is the vector space of skew symmetric $n \times n$ matrices — that is, A satisfying $A^t = -A$.

10. Show that the set of $m \times n$ matrices of rank r is a submanifold of $\mathbb{R}^{mn}$ of codimension $(m-r)(n-r)$. (Hint: Suppose, for simplicity, that an $m \times n$ matrix A has the form $A = \begin{pmatrix} B_{r\times r} & C \\ D & E \end{pmatrix}$, where the $r \times r$ matrix B is nonsingular. Postmultiply by the non singular matrix $\begin{pmatrix} I & -B^{-1}C \\ 0 & I \end{pmatrix}$ to prove that $\operatorname{rank}(A) = r$ if and only if $E - DB^{-1}C = 0$.)