微分拓扑, 2018 年春季

作业 4

上交时间及方式：2017.06.05

1. (a) Write down the orientation of S^2 (as the boundary of B^3) by writing a positively oriented ordered basis for the tangent space at each (a, b, c).

   (b) Let $f : S^2 \to \mathbb{R}$ be defined by $f(a, b, c) = c$. For the regular values $t \in (-1, 1)$, $f^{-1}(t)$ is the longitude circe at height t. Explicitly exhibit a positively oriented vector at a typical point of $f^{-1}(t)$.

2. Let X and Z be transversal submanifolds in Y, all three being oriented. Let $X \cap Z$ denote the intersection manifold with the orientation prescribed by the inclusion map $i : X \hookrightarrow Y$. Show that the two orientations on the intersection manifold are related by

$$X \cap Z = (-1)^{(\text{codim } X)(\text{codim } Z)} Z \cap X.$$

   (HINT: Show that the orientation of $S = X \cap Z$ is specified by the formula

$$\left[N_y(S; X) \oplus N_y(S; Z)\right] \oplus T_y(S) = T_y(Y)$$

   And for $Z \cap X$, the first two spaces are interchanged.)

3. Let X be an orientable manifold. Show that the product orientation on $X \times X$ is the same for all choices of orientation on X.

   (HINT: Show this for a vector space.)

4. Let

$$p(z) = z^m + a_1 z^{m-1} + \cdots + a_m$$

   be a complex polynomial function. Let r be a positive real number such that

$$r^m > |a_1| \, r^{m-1} + \cdots + |a_m|$$

   Show that p has a root inside the disk $\{|z| < r\}$ of radius r.

5. Show that the Euler characteristic of an orientable manifold is the same for all choices of orientation.

6. Prove the following are equivalent conditions:

   (a) x is a Lefschetz fixed point of $f : X \to X$.

   (b) 0 is a Lefschetz fixed point of $df_x : T_x(X) \to T_x(X)$.

   (c) $df_x : T_x(X) \to T_x(X)$ is a Lefschetz map.

7. (a) Show that the map $z \longmapsto z + z^m$ $(m > 0)$ has a fixed point with local Lefschetz number m at the origin of $\mathbb{C}$.

   (b) Show that for any $c \neq 0$, the homotopic map $z \longmapsto z \to z + z^m + c$ is Lefschetz, with m fixed points that are all close to zero if c is sufficiently small.

   (c) SHow that the map $z \longmapsto z + (\overline{z})^m$ $(m > 0)$ has a fixed point with local Lefschetz number $-m$ at the origin of $\mathbb{C}$

8. Recall that a vector field $\vec{v}$ on a manifold X in $\mathbb{R}^N$ is a particular type of map $\vec{v} : X \to \mathbb{R}^N$. Show that at a zero x, the derivative $d\vec{v}_x : T_x(X) \to \mathbb{R}^N$ actually carries $T_x(X)$ into itself.

   HINT: If $X = \mathbb{R}^k \times \{0\}$, then the claim is obvious. But you can always reduce to this case by suitable local parametrization around x in $\mathbb{R}^N$