微分拓扑, 2018 年春季

作业 3

上交时间及方式：2017.04.24

REQUIRED:

1. Suppose that X is a manifold with boundary and $x \in \partial X$. Let $\phi : U \to X$ be a local parametrization with $\phi(0) = x$. Then $d\phi_0 : \mathbb{R}^k \to T_x(X)$ is an isomorphism. Let $H_x(X)$ be the image of H^k under $d\phi_0$, that is $H_x(X) = d\phi_0(H^k)$. It can be shown that $H_x(X)$ does not depend on the choice of local parametrization.

   (a) Show that there are precisely two unit vectors in $T_x(X)$ that are perpendicular to $T_x(\partial X)$: one lies inside $H_x(X)$ while the other outside.

   (b) The one outside of $H_x(X)$ is called the *outward unit normal vector*, denoted by $\vec{n}(x)$. If X sits in $\mathbb{R}^N$, then $\vec{n}$ may be considered to be a map of ∂X into $\mathbb{R}^N$. Show that $\vec{n}$ is a smooth map.

2. Prove the following theorem of Frobenius: If the entries in an $n \times n$ matrix A are all nonnegative, then A has a real nonnegative eigenvalue.

   (HINT: It suffices to show that A is non singular, since otherwise 0 is an eigenvalue. A can be considered to be a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$. Consider the map $v \mapsto (Av)/|Av|$ restricted to $S^{n-1} \to S^{n-1}$. Show that this maps the "first quadrant"

   $$Q = \left\{(x_1, \cdots, x_n) \in S^{n-1} \;\middle|\; x_i \geq 0\right\}$$

   into itself. It can be shown that Q is homeomorphic to B^{n-1}. Now apply the previous exercise.)

3. (For the ε-Neighborhood Theorem) Let Y be a submanifold in $\mathbb{R}^M$. Let $\mathbb{R}^+$ be the set of positive real numbers and $\varepsilon : Y \to \mathbb{R}^+$ be a smooth function. Define the open set Y^ε by

   $$Y^\varepsilon = \left\{x \in \mathbb{R}^M \;\middle|\; d(x, y) < \varepsilon(y) \text{ for some } y \in Y\right\}.$$

   Show that any neighborhood $\widetilde{U}$ of Y contains some Y^ε; moreover, if Y is compact, ε may be taken constant.

   (HINT: partition of unity.)

4. (General Position Lemma) Let X and Y be submanifolds of $\mathbb{R}^N$. Show that for almost every $a \in \mathbb{R}^N$, the translate $X + a$ intersects Y transversally.

5. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map, $n > 1$, and let $K \subset \mathbb{R}^n$ be compact and $\varepsilon > 0$. Show that there exists a map $f' : \mathbb{R}^n \to \mathbb{R}^n$ such that df'_x is never zero, but $|f - f'| < \varepsilon$ on K. Prove that this result is false for $n = 1$.

   (HINT: Let $M(n)$ be the set of $n \times n$ matrices, and show that the map $F : \mathbb{R}^n \times M(n) \to M(n)$, defined by $f(x, A) = df_x + A$, is a submersion. Pick A so that F_a transverse to $\{0\}$; where is $n > 1$ used?)

6. Let $f : \mathbb{R}^k \to \mathbb{R}^k$ be a smooth function. For any $a \in \mathbb{R}^k$, define

   $$f_a(x) = f(x) + a_1 x_1 + \cdots + a_k x_k$$

   Show that for almost all $a \in \mathbb{R}^k$, f_a is a Morse function.

7. Let Δ be the diagonal in $X \times X$. Show that the orthogonal complement to $\mathcal{T}_{(x,x)}\Delta$ in $\mathcal{T}_{(x,x)}(X \times X)$ is the collection of vectors $\left\{(v, -v) \;\middle|\; v \in \mathcal{T}_x X\right\}$.