1. Note that

$$\begin{cases} T_1^2+T_2^2+T_3^2=0 \\ T_1^2-T_2^2-T_3^2+1=0 \end{cases} \quad \text{and} \quad \begin{cases} T_1^2+T_2^2+T_3^2=0 \\ 2T_1^2+1=0 \end{cases}$$

have the same solutions. If $\mathrm{char}(k) \neq 2$ and $\sqrt{-2} \notin k$, then the affine algebraic k-set defined by the equations $T_1^2+T_2^2+T_3^2=0$ and $T_1^2-T_2^2-T_3^2+1=0$ is irreducible. If $\sqrt{-2} \in k$, then the affine algebraic k-set has two irreducible components

$$\begin{cases} T_2^2+T_3^2=-\frac{1}{2} \\ T_1=-\frac{\sqrt{-2}}{2} \end{cases} \quad \text{and} \quad \begin{cases} T_2^2+T_3^2=-\frac{1}{2} \\ T_1=+\frac{\sqrt{-2}}{2} \end{cases}$$

2. The set of zero points of the equation

$$\begin{cases} T_2^2-T_1T_3=0 \\ T_1^2-T_2^3=0 \end{cases}$$

is denoted by $V(T_2^2-T_1T_3, T_1^2-T_2^3)$. We have

$$V(T_2^2-T_1T_3T_2, T_1^2-T_2^3) = V(T_2, T_1^2-T_2^3) \cup V(T_2^2-T_1T_3, T_1^2-T_2^3)$$

In fact $V(T_2, T_1^2-T_2^3)$ has affine coordinate ring

$$k[T_1,T_2,T_3]/(T_2, T_1^2-T_2^3) \cong k[T_3,T_1]/(T_1^2)\,.$$

So the irreducible component is birationally isomorphic to the affine plane

$$\begin{aligned} V(T_2^3-T_1T_3T_2, T_1^2-T_2^3) &= V(T_1^2-T_1T_2T_3, T_1^2-T_2^3) \\ &= V(T_1, T_1^2-T_2^3) \cup V(T_1-T_2T_3, T_1^2-T_2^3) \\ &= V(T_1, T_1^2-T_2^3) \cup V(T_1-T_2T_3, T_2^2(T_3-T_2)) \\ &= V(T_1, T_1^2-T_2^3) \cup V(T_1-T_2T_3, T_2^2) \cup V(T_1-T_2T_3, T_3-T_2) \end{aligned}$$

In fact:

$$k[T_1,T_2,T_3]/(T_1, T_1^2-T_2^3)$$
$$k[T_1,T_2,T_3]/(T_1-T_2T_3, T_2) \cong k[T_1,T_3]/(T_1) \cong k[T_3]$$
$$k[T_1,T_2,T_3]/(T_1-T_2T_3, T_3-T_2) \cong k[T_1,T_2]/(T_1-T_2^2)$$

is birationally isomorphic to $\mathbf{A}_k^1$.

5.

$$\because T_1^3+T_2^3=1$$
$$\therefore T_1+T_2 \neq 0$$

Let $x=\frac{1}{T_1+T_2}$ and $y=\frac{3(T_1-T_2)}{T_1+T_2}$

$$\therefore T_1+T_2=\frac{1}{x} \qquad T_1-T_2=\frac{y}{3x}$$
$$\therefore T_1=\frac{3+y}{6x} \qquad T_2=\frac{3-y}{6x}$$
$$\therefore T_1^3+T_2^3=\frac{(3+y)^3+(3-y)^3}{6^3x^3}=1$$
$$\therefore 2\times 3^3+2\times 3\times 3y^2=6^3x^3$$
$$\text{i.e.}\quad y^2=2^2\cdot 3x^3-3$$

Let $y' = \frac{y}{3\cdot 2}$, $x' = x$.

$$\therefore y' = \frac{x^3}{3} - \frac{1}{12}$$

Hence $T_1^3 + T_2^3 - 1 = 0$ and $T_1^2 - \frac{T_2^3}{3} + \frac{1}{12}$ are birationally isomorphic.

4. Let $T_i = t_i T_1,\ i = 2, 3, \ldots, n$. Then

$$\begin{aligned} F(T_1, T_2, \ldots, T_n) &= F(T_1, t_2 T_1, \ldots, t_n T_1) \\ &= G(T_1, t_2 T_1, \ldots, t_n T_1) + H(T_1, t_2 T_1, \ldots, t_n T_1) \\ &= T_1^d G(1, t_2, \ldots, t_n) + T_1^{d-1} H(1, t_2, \ldots, t_n) \end{aligned}$$

If $T_1 \neq 0$, then

$$T_1 G(1, t_2, \ldots, t_n) + H(1, t_2, \ldots, t_n) = 0$$

$$\therefore T_1 = \frac{H(1, t_2, \ldots, t_n)}{G(1, t_2, \ldots, t_n)}$$

So there is a birational map

$$V(F) \cap (\mathbf{A}_k^n - V(T_1)) \to \mathbf{A}_k^{n-1}, \qquad (T_1, \ldots, T_n) \mapsto \Big(\frac{T_2}{T_1}, \ldots, \frac{T_n}{T_1}\Big)$$

Its inverse map is

$$\mathbf{A}_k^{n-1} - V(G(1, t_2, \ldots, t_n)) \to V(F)$$
$$(t_2, t_3, \ldots, t_n) \mapsto \Big(\frac{H(1, t_2, \ldots, t_n)}{G(1, t_2, \ldots, t_n)}, t_2 \frac{H}{G}, \ldots, t_n \frac{H}{G}\Big)$$