§1

1. Example 1: Take $n = 1$ and $\mathrm{char}(k) = 0$, then $\mathbb{Q} \subset k$. Then the functor $\mathbb{Z}\colon \mathscr{A}lg_k \to \mathscr{S}et$ defined by $\mathbb{Z}(A) = \mathbb{Z}$ is a constant functor. It is a subfunctor of $\mathscr{A}_k^1$, but $\mathbb{Z}$ is not an affine variety. In fact if $\mathbb{Z}$ is an affine variety, then there exists an integer $n \geqslant 1$ and an ideal I of $k[T_1, T_2, \ldots, T_n]$ such that

$$\mathbb{Z}(A) = \mathrm{Hom}_k(k[T_1, T_2, \ldots, T_n]/I, A)$$

for all $A \in \mathscr{A}lg_k$. Taking $A = k[T_1, T_2, \ldots, T_n]/I$, we have

$$\mathbb{Z}(A) = \mathrm{Hom}_k(k[T_1, T_2, \ldots, T_n]/I, k[T_1, T_2, \ldots, T_n]/I) \neq \mathbb{Z})\,.$$

2. If $n = 1$, then $(S) = (S')$. So S and S' define the same algebraic varieties.

If $n = 2$, we have

$$\sigma_1(x_1, x_x) = T_1 + T_2 \in S'$$
$$S \cap S' \ni \sigma_1(T_1T_2)^2 = x_1^2 + x_2^2 - 2x_1x_2$$
$$\therefore x_1^2 + x_2^2 \in (S) \cap (S') \Longleftrightarrow 2x_1x_2 \in (S) \cap (S')$$
$$\therefore \mathrm{char}(k) \neq 2, \quad (S) = (S')$$

By induction, we can prove that if $\mathrm{char}(k) > n - 1$, then $(S) = (S')$.

3. Note that

$$T_2^2T_1^2 + T_1^2 + T_2^3 + T_2 + T_1T_2 = T_2^2(T_1^2 + T_2) + (T_1^2 + T_2) + T_2T_1 \in (T_1^2 + T_2, T_1)$$

and

$$T_2T_1^2 + T_2^2 + T_1 = T_2(T_1^2 + T_2) + T_1 \in (T_1^2 + T_2, T_1)$$

So we have

$$(T_2^2T_1^2 + T_1^2 + T_2^3 + T_2 + T_1T_2, T_2T_1^2 + T_2^2 + T_1) \subset (T_1^2 + T_2, T_1)$$

Conversely, we have

$$T_2^2T_1^2 + T_1^2 + T_2^3 + T_2 + T_1T_2 - (T_2T_1^2 + T_2^2 + T_1)T_2 = T_1^2 + T_2 \in (T_2^2T_1^2 + T_1^2 + T_2^3 + T_2 + T_1T_2, T_2T_1^2 + T_2^2 + T$$
$$T_1 = (T_2T_1^2 + T_2^2 + T_1) - T(T_1^2 + T_2) \in (T_2^2T_1^2 + T_1^2 + T_2^3 + T_2 + T_1T_2, T_2T_1^2 + T_2^2 + T_1)$$

4. If $X \subset \mathbf{A}_k^n$ and $X' \subset \mathbf{A}_k^m$ are two varieties, then there exists two ideals $I(X) \subset k[x_1, x_2, \ldots, x_n]$ and $I(X') \subset k[y_1, y_2, \ldots, y_m]$ such that

$$X(K) = \mathrm{Hom}_k(k[x_1, x_2, \ldots, x_n]/I(X), K)$$
$$X'(K) = \mathrm{Hom}_k(k[y_1, y_2, \ldots, y_m]/I(X'), K)$$

Thus

$$X(K) \times X'(K) = \mathrm{Hom}_k(k[x_1, x_2, \ldots, x_n]/I(X), K) \times \mathrm{Hom}_k(k[y_1, y_2, \ldots, y_m]/I(X'), K)$$

So we have only to prove that there exists a bijective of

$$X(K) \times X'(K) = \mathrm{Hom}_k(k[x_1, x_2, \ldots, x_n]/I(X), K) \times \mathrm{Hom}_k(k[y_1, y_2, \ldots, y_m]/I(X'), K)$$

to

$$\mathrm{Hom}_k(k[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m]/(I(X), I(X')), K)\,,$$

i.e.,

$$X(K) \times X'(K) = \mathrm{Sol}(I(X) \cup I(X'), K)\,.$$

5.Since X and X' are two subvarieties of $\mathbf{A}_k^n$, there exists ideals $I(X), I(X') \subset k[x_1, x_2, \ldots, x_n]$ such that

$$X(K) = \mathrm{Hom}_k(k[x_1, x_2, \ldots, x_n]/I(X), K) = \mathrm{Sol}(I(X), K)$$
$$X'(K) = \mathrm{Hom}_k(k[x_1, x_2, \ldots, x_n]/I(X'), K) = \mathrm{Sol}(I(X'), K)$$

$$\begin{aligned} X(K) \cap X'(K) &= \mathrm{Sol}(I(X), K) \cap \mathrm{Sol}(I(X'), K) \\ &= \mathrm{Sol}(I(X) \cup I(X'), K) \\ &= \mathrm{Hom}_k(k[x_1, x_2, \ldots, x_n]/(I(X), I(X')), K) \end{aligned}$$

but the functor $K \to X(K) \cup X'(K)$ is not a subvariety of $\mathbf{A}_k^n$:

For example take $k = \mathbb{Q}$, $n = 1$, let X be the affine variety associated to the equation $x = 1$, and X' be the variety defined by $x = 0$. If $(f(x))$ is the ideal defined by the functor $K \to X(K) \cup X'(K)$, then

$$X(K) \cup X'(K) = \mathrm{Hom}_{\mathbb{Q}}(k[x]/(f(x)), K)$$

Now we let $K = \mathbb{Q} \times \mathbb{Q}$ be a $\mathbb{Q}$-algebra. Then

$$\begin{aligned} X(K) \cup X'(K) &= \mathrm{Hom}_{\mathbb{Q}}(k[x]/(f(x)), \mathbb{Q} \times \mathbb{Q}) \\ &= \mathrm{Hom}_{\mathbb{Q}}(k[x]/(f(x)), \mathbb{Q}) \times \mathrm{Hom}_{\mathbb{Q}}(k[x]/(f(x)), \mathbb{Q}) \\ &= (X(\mathbb{Q}) \cup X'(\mathbb{Q})) \times (X(\mathbb{Q}) \cup X'(\mathbb{Q})) \\ &= \{0, 1\} \times \{0, 1\} \\ &= \{(0, 0), (0, 1), (1, 0), (1, 1)\} \end{aligned}$$

But

$$X(K) = \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}[x]/(x), \mathbb{Q}) = \{(0, 0)\}$$
$$X'(K) = \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}[x]/(x-1), \mathbb{Q} \times \mathbb{Q}) = \{(1, 1)\}$$

a contradiction.

6. $\Longrightarrow$ is easy.

$\Longleftarrow$. Note that

$$\mathrm{Sol}(S, k) = \mathrm{Hom}_k(k[T]/(S), k)$$
$$\mathrm{Sol}(S', k) = \mathrm{Hom}_k(k[T]/(S'), k)$$

Since S, S' are two system of linear equations over k and $\mathrm{Sol}(S, k) = \mathrm{Sol}(S', k)$, there is an invertible linear transformation L such that $S = L(S')$. So $(S) = (S')$. (This is the linear algebra).

§2

1. Since $T_1^2 = T_2^3$ in A, we have $T_1^2/T_2^2 = T_2$. Hence T_1/T_2 is contained in the quotient field of A. But $T_1/T_2 \neq A$. T_1/T_2 is integral over A, since T_1/T_2 satisfies the equation $x^2 - T_2 = 0$.

6. We have

$$\overline{\{(z_1, z_2) \in \mathbb{C} \mid |z_1|^2 + |z_2|^2 = 1\}}_{\mathrm{Zar}} = \mathbb{C}^2$$

Let $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y + 2i$. Then the equation $|z_1|^2 + |z_2|^2 = 1$ becomes

$$x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1\,,$$

which is a 3-dimensional sphere. But any polynomial $f(z_1, z_2)$ is equivalent to a system of polynomials

$$\begin{cases} P(x_1, y_1, x_2, y_2) & = 0 \\ Q(x_1, y_1, x_2, y_2) & = 0 \end{cases}$$

which is a 2-dimensional geometrical object. So $\overline{|z_1|^2 + |z_2|^2 = 1}_{\text{Zar}}$ is not contained in the proper algebraic set of $\mathbb{C}^2$.

§3

2. In fact the coordinate ring of the variety defined by the polynomial $xy - 1 = 0$ is $k[x, y]/(xy - 1) \cong k[x, \frac{1}{x}] \ncong k[x]$.

3. $\mathrm{Im}(f) = \mathbf{A}^2_k(K) - \{(0, y) \mid y \neq 0\}$ open dense.

4. $\mathrm{Hom}_k(k[x], k[x]) \cong K^*$.

5. Note that

$$\begin{aligned} \mathcal{O}(X) &\cong k[T]/I(X) \\ \mathcal{O}(Y) &\cong k[T]/I(Y) \\ X &= \mathrm{Hom}_k(k[T]/I(X), \bullet) \\ Y &= \mathrm{Hom}_k(k[T]/I(Y), \bullet) \\ X \times Y &= \mathrm{Hom}_k(k[T]/I(X) \otimes_k k[T]/I(Y), \bullet) \\ \therefore \mathcal{O}(X \times_k Y) &\cong \mathcal{O}(X) \otimes_k \mathcal{O}(Y) \end{aligned}$$

6. Let $k[\mu_{ij}]$ be a polynomial ring with n^2 indeterminates $\{\mu_{ij} \mid i, j = 1, 2, \ldots, n\}$. Then

$$K \to \mathrm{GL}_n(K) = \mathrm{Hom}_k(k[\mu_{ij}, T]/(\det(\mu_{ij})T - 1), K)$$

It is enough to prove that for any $A = (a_{ij}) \in \mathrm{GL}_n(K)$, we can define a homomorphism

$$\varphi_A \colon k[\mu_{ij}, T] \to K, \qquad \mu_{ij} \mapsto a_{ij} \text{ and } T \mapsto \det(A)^{-1},$$

which induces a homomorphism

$$\varphi \colon k[\mu_{ij}, T]/(\det(\mu_{ij})T - 1) \to K$$

Conversely we have a matrix $(\varphi(T_{ij}))$ such that $\det(\varphi(T_{ij}))$.

$$\therefore \varphi(T) \Longrightarrow (\varphi(T_{ij})) \in \mathrm{GL}_n(K)$$