MATH 251-019: Homework 13 (Due: 12/06/2017)

Please make your hand-writing clear to read. Please box your final answer.

1. (Fall 2016, Final, Question 3) Consider the autonomous equation

$$y' = y^2(y-2)(y-3).$$

Suppose $y(t)$ is a solution and $y(3) = 1$, then what is the limit of $y(t)$ as $t \to \infty$?

(A) 0
(B) 1
(C) 2
(D) 3

2. (Fall 2014, Final, Question 6) Find the Laplace transform $\mathcal{L}\{u_3(t)(t^2-2t+1)\}$.

(A) $F(s) = e^{-3s}\dfrac{16s^2-8s+2}{s^3}$
(B) $F(s) = e^{-3s}\dfrac{-2s^2-2s+2}{s^3}$
(C) $F(s) = e^{-3s}\dfrac{s^2-2s+2}{s^3}$
(D) $F(s) = e^{-3s}\dfrac{4s^2+4s+2}{s^3}$

3. (Fall 2014, Final, Question 7) Find the inverse Laplace transform $\mathcal{L}^{-1}\{e^{-s}\frac{4}{(s^2+4s+3)(s+1)}\}$.

(A) $f(t) = u_1(t)(e^{-3t+3} - e^{-t+1} + 2e^{-t+1}(t-1))$
(B) $f(t) = u_1(t)(e^{-3t-3} + 2e^{-t-1}(t+1))$
(C) $f(t) = u_1(t)(e^{-3t} + 2e^{-t}t)$
(D) $f(t) = \delta(t-1)(e^{-3t} - e^{-t} + 2e^{-t}t)$

4. (Fall 2016, Final, Question 7) Given that the point $(-2, 3)$ is a critical point of the nonlinear system of equations

$$x' = xy - x + 2y - 2$$
$$y' = y^2 - 5y + 6$$

The critical point $(-2, 3)$ is an

(A) unstable node.
(B) unstable saddle point.
(C) asymptotically stable spiral point.
(D) asymptotically improper node.

5. (Spring 2013, Exam 2, Question 10) Find the general solution of the system of linear equations

$$\mathbf{x}' = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$$

(Hint: The critical point is an improper node.)

6. (Fall 2015, Final, Question 17) Suppose the displacement $u(x,t)$ of a piece of flexible string is given by the initial-boundary value problem

$$\begin{aligned} &4u_{xx} = u_{tt}, \qquad 0 < x < 3, \quad t > 0 \\ &u(0,t) = 0, \quad u(3,t) = 0, \\ &u(x,0) = 0, \\ &u_t(x,0) = 5\cos(\pi x) + 2. \end{aligned}$$

   (a) TRUE or FALSE: At $t = 0$, the string is at rest (i.e., having zero initial velocity).

   (b) When $t = 0$, what is the displacement of the string at the midpoint, $x = \frac{3}{2}$?

   (c) In what specific form will be the general solution appear?

   (1) $u(x,t) = \sum_{n=1}^{\infty} C_n \cos(\frac{2n\pi t}{3}) \sin(\frac{n\pi x}{3})$ (2) $u(x,t) = \sum_{n=1}^{\infty} C_n \sin(\frac{2n\pi t}{3}) \cos(\frac{n\pi x}{3})$

   (3) $u(x,t) = \sum_{n=1}^{\infty} C_n \cos(\frac{2n\pi t}{3}) \cos(\frac{n\pi x}{3})$ (4) $u(x,t) = \sum_{n=1}^{\infty} C_n \sin(\frac{2n\pi t}{3}) \sin(\frac{n\pi x}{3})$

   (d) TRUE or FALSE: The coefficients of the solution in part (c) above can be found using the integral

$$C_n = \frac{1}{n\pi} \int_0^3 (5\cos(\pi x) + 2) \sin(\frac{n\pi x}{3})\, dx.$$

   (e) TRUE or FALSE: The boundary conditions indicate that the string is securely fixed and held motionless at both ends.

   (f) TRUE or FALSE: The quantity $u_x(2,5)$ represents the velocity of the string when $t = 5$ at the point $x = 2$.

7. (Fall 2016, Final, Question 15) Suppose the displacement $u(x,t)$ of a piece of flexible string is given by the initial boundary value problem

$$\begin{aligned} &49u_{xx} = u_{tt}, \qquad 0 < x < 1, \quad t > 0 \\ &u(0,t) = 0, \quad u(1,t) = 0, \\ &u(x,0) = x\cos(\frac{\pi x}{2}), \\ &u_t(x,0) = 0. \end{aligned}$$

   (a) Given a physical interpretation (of functions in the problem): __________ is the displacement and __________ is the velocity of the flexible string at point x at time t.

(b) Circle the correct answer regarding the meaning of the boundary conditions.

   i. They guarantee the displacement is 0 at the endpoints of the string.
   ii. They keep motion of the string free and undamped.

(c) In what specific form will the general solution appear?

$$(1)\ u(x,t) = \sum_{n=1}^{\infty} C_n \sin(7n\pi t)\sin(n\pi x) \quad (2)\ u(x,t) = \sum_{n=1}^{\infty} C_n \cos(7n\pi t)\sin(n\pi x)$$

$$(3)\ u(x,t) = \sum_{n=1}^{\infty} C_n \sin(7n\pi t)\cos(n\pi x) \quad (4)\ u(x,t) = \sum_{n=1}^{\infty} C_n \cos(7n\pi t)\cos(n\pi x)$$

(d) TRUE or FALSE: The coefficients of the solution in part (c) above can be found using the integral

$$C_n = 2\int_0^1 x\cos(\frac{\pi x}{2})\cos(n\pi x)\ dx.$$

(e) Use the boundary conditions to determine the steady-state displacement $v(x)$ of this string.

(f) How will you modify the initial-boundary value problem stated above (by making one change to the problem) so that its solution is equal to the steady-state displacement for all $t > 0$?

   i. Add a damping term into the equation.
   ii. Set $u(x,0) = 0$.
   iii. Set $u_t(x,0) = v'(x)$.
   iv. Any of the above changes would do.