MATH 251-019: Homework 12 (Due: 11/29/2017)

Please make your hand-writing clear to read. Please $\boxed{\text{box}}$ your final answer.

1. (Fall 2016, Final, Question 9) Find the steady-state solution, $v(x)$, of the heat conduction problem with nonhomogeneous boundary conditions

$$\begin{aligned} &\alpha^2 u_{xx} = u_t, \qquad 0 < x < 9, \quad t > 0 \\ &u(0,t) - u_x(0,t) = 0, \qquad u(9,t) - u_x(9,t) = 18, \\ &u(x,0) = f(x). \end{aligned}$$

(A) $v(x) = 2$
(B) $v(x) = 2 + 2x$
(C) $v(x) = 2 - 2x$
(D) $v(x) = 2x$

2. (Spring 2015, Final, Question 9) Consider the third order linear partial differential equation

$$u_{tt} + u_{xtt} = u_{xx}.$$

Use the substitution $u(x,t) = X(x)T(t)$, where $u(x,t)$ is not the trivial solution, which of following ordinary differential equation pairs does it separate into? Please use $-\lambda$ as the separation constant.

(A) $T'' + \lambda T = 0$, $X'' + \lambda(X + X') = 0$.
(B) $T'' - \lambda T = 0$, $X'' - \lambda(X - X') = 0$.
(C) $T'' - \lambda T = 0$, $X'' + \lambda(X + X') = 0$.
(D) $T'' + \lambda T = 0$, $X'' + X - \lambda X' = 0$.

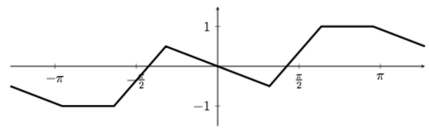
3. (Spring 2017, Final, Question 5) Suppose $y_1(t)$ and $y_2(t)$ are any two solutions of the second order linear equation

$$(1 + t^2)y'' + 4ty' + (1 - t)y = 0.$$

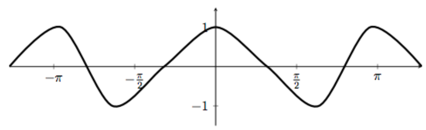
Which function below can possibly be their Wronskian, $W(y_1, y_2)(t)$?

(A) $W(y_1, y_2) = \dfrac{-5}{(1 + t^2)^2}$
(B) $W(y_1, y_2) = -4\arctan(t)$
(C) $W(y_1, y_2) = 7e^{-2t^2}$
(D) $W(y_1, y_2) = 2(1 + t^2)^2$

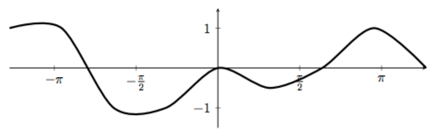
4. (Fall 2014, Final, Question 12) Each graph below shows a single period of certain periodic function. Which function will have a Fourier series that **does not** contain any sine term?



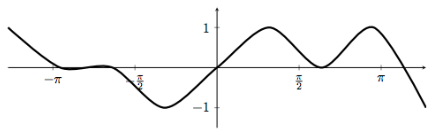
(A)



(B)



(C)



(D)

5. Determine whether each statement below is **TRUE or FALSE**. You must justify your answers.

   (a) (Spring 2017, Final, Question 11(d)) The Fourier series representing any odd periodic function $f(x)$ necessarily converges to 0 at $x = 0$.

   (b) (Spring 2015, Final, Question 12(b)) Using the formula $u(x, t) = X(t)T(t)$, the partial differential equation $u_t + \sin(x)u_{xxx} = 0$ can be separated into 2 ordinary differential equations.

   (c) (Spring 2015, Final, Question 12(c)) Every even periodic function has a Fourier series containing a non-zero constant term $\frac{a_0}{2}$.

   (d) (Spring 2015, Final, Question 12(d)) Any odd periodic function has a Fourier series consisting only of sine functions.

   (e) (Fall 2014, Finial, Question 13(d)) The constant term in the Fourier series representing an odd periodic function may be nonzero.

6. (Fall 2016, Final, Question 13) Let $f(x) = 4 - x^2$, $0 < x < 2$

   (a) Consider the **odd** periodic extension, of period $T = 4$, of $f(x)$. Sketch 3 periods, on the interval $-6 < x < 6$, of this function.

   (b) To what value does the Fourier series of this odd periodic extension converge at $x = -1$? At $x = 8$?

   (c) Consider the **even** periodic extension, of period $T = 4$, of $f(x)$. Sketch 3 periods, on the interval $-6 < x < 6$, of this function.

   (d) Find $\frac{a_0}{2}$, the constant term of the Fourier series of the even periodic function described in (c).

(e) State TRUE/FALSE with reason. For the even periodic function in part (c), the Fourier cosine coefficients a_n, $n \geq 1$, are given by

$$a_n = \frac{1}{2} \int_{-2}^{2} (4 - x^2) \cos(\frac{n\pi x}{2})\, dx.$$

7. Suppose the temperature distribution function $u(x,t)$ of a rod is given by the initial-boundary value problem

$$\begin{aligned} &5u_{xx} = u_t, \qquad 0 < x < \pi, \quad t > 0 \\ &u_x(0,t) = 0, \qquad u_x(\pi,t) = 0, \\ &u(x,0) = 1 - 5\cos(2x) + \cos(3x). \end{aligned}$$

(a) State the general form of its solution. Then find the particular solution of the initial-boundary value problem.

(b) Find $\lim_{t\to\infty} u(x,t)$, $0 < x < \pi$. What is the physical interpretation of this limit?

(c) Suppose the **initial condition** was changed to $2 - 100\cos(x)$. What is $\lim_{t\to\infty} u(\frac{\pi}{2}, t)$ in this case?