MATH 251-019: Homework 11 (Due: 11/15/2017)

Please make your hand-writing clear to read. Please $\boxed{\text{box}}$ your final answer.

1. (Fall 2016, Final, Question 1) Consider the initial/boundary value problems below. Which one is certain to have a unique solution for every value of α and β?

$$\textbf{I} \qquad y'' + 16y = 0, \quad y(\alpha) = \alpha, \quad y'(\alpha) = \beta.$$
$$\textbf{II} \qquad y'' + 16y = 0, \quad y(0) = \alpha, \quad y(\beta) = 0.$$

   (A) Both **I** and **II**.
   (B) **I** only.
   (C) **II** only.
   (D) Neither.

2. (Spring 2016, Final, Question 10) Use the substitution $u(x,t) = X(x)T(t)$, where $u(x,t)$ is not the trivial solution, consider the two statements below.
   (1) The equation $5u_{xx} = xu_x + \frac{1}{x}u_{tt}$ can be separated into two ordinary differential equations.
   (2) The boundary condition $u(0,t) = 0$ and $u_x(\pi,t) = 0$ can be rewritten into $X(0) = 0$ and $X(\pi) = 0$.
   What can you say regarding the truthfulness of these statements?

   (A) Only (1) is true.
   (B) Only (2) is true.
   (C) Both are true.
   (D) Neither is true.

3. (Fall 2016, Final, Question 8) Consider the two linear partial differential equations.

$$\textbf{I} \qquad x^5t^3e^{-x}u_{xx} - xt^3u_x + x^2t^6u_{tt} = 0$$
$$\textbf{II} \qquad u_{xx} - 2u_{tx} - 4u_{tt} = 0$$

   Use the substitution $u(x,t) = X(x)T(t)$, where $u(x,t)$ is not the trivial solution, and attempt to separate each equation into two ordinary differential equations. Which statement below is true?

   (A) Neither equation is separable.
   (B) Only **I** is separable.
   (C) Only **II** is separable.
   (D) Both equations are separable.

4. Determine whether each statement below is **TRUE or FALSE**. You must justify your answers.

(a) (Fall 2016, Final, Question 11(e)) Using the formula $u(x,t) = X(x)T(t)$, the boundary conditions $u(0,t) = 1$ and $u_x(\pi, t) = 0$ can be rewritten as $X(0) = 1$ and $X'(\pi) = 0$.

(b) (Fall 2016, Final, Question 11(f)) Consider the heat conduction problem $u_t = \alpha^2 u_{xx}$, $0 < x < 3$, $t > 0$. The boundary conditions $u_x(0,t) = 0$ and $u_x(3,t) = 3$ mean that the left of the bar is perfectly insulated and the right end is not.

(c) (Spring 2017, Final, Question 11(f)) $X(x) = 20$ is an eigenfunction of the 2-point boundary value problem

$$X'' + \lambda X = 0, \qquad X'(0) = 0, \qquad X'(1) = 0.$$

5. (Fall 2016, Final, Question 12) Consider the two-point boundary value problem.

$$X'' + \lambda X = 0, \qquad X'(0) = 0, \quad X'(\pi) = 0.$$

(a) Find all **positive** eigenvalues and corresponding eigenfunctions of the boundary value problem.

(b) Is $\lambda = 0$ an eigenvalue of this problem? If yes, find its corresponding eigenfunction. If no, briefly explain why it is not an eigenvalue.

6. (Spring 2016, Final, Question 14) Consider the two-point boundary value problem.

$$X'' + \lambda X = 0, \qquad X'(0) = 0, \quad X(5) = 0.$$

(a) Find all **positive** eigenvalues and corresponding eigenfunctions of the boundary value problem.

(b) Is $\lambda = 0$ an eigenvalue of this problem? If yes, find its corresponding eigenfunction. If no, briefly explain why it is not an eigenvalue.

7. (Spring 2016, Final, Question 16) Suppose the temperature distribution function $u(x,t)$ of a rod is given by the initial-boundary value problem

$$\begin{aligned} &5u_{xx} = u_t, \qquad 0 < x < \pi, \quad t > 0 \\ &u(0,t) = 0, \qquad u(\pi, t) = 0, \\ &u(x,0) = -5\sin(2x) + \sin(3x) \end{aligned}$$

(a) State the general form of its solution. Then find the particular solution of the initial-boundary value problem.

(b) Find $\lim_{t\to\infty} u(x,t)$, $0 < x < \pi$. What is the physical interpretation of this limit?

(c) Suppose the **boundary conditions** were changed to $u(0,t) = 1$, $u(\pi, t) = 5$. What is $\lim_{t\to\infty} u(\frac{\pi}{2}, t)$ in this case?