MATH 251-019: Homework 10 (Due: 11/02/2017)

Please make your hand-writing clear to read. Please $\boxed{\text{box}}$ your final answer.

1. (Spring 2017, Exam 2, Question 5) Which ordinary differential equation below is equivalent to the following system of linear equations?

$$\begin{cases} x_1' = -x_2 \\ x_2' = 3x_1 - x_2 + \cos t \end{cases}$$

(A) $u'' - 3u' + u = \cos t$
(B) $u'' + 3u' + u = -\cos t$
(C) $u'' + u' + 3u = -\cos t$
(D) $u'' + u' - 3u = \cos t$

2. (Spring 2017, Exam 2, Question 6) Consider a certain system of linear equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a 2×2 matrix of real numbers. Suppose one of the eigenvalues of the coefficient matrix $\mathbf{A}$ is $r = 2 - i$, which has a corresponding eigenvector $\begin{bmatrix} 1 \\ i \end{bmatrix}$. What is the system's real-valued general solution?

(A) $\mathbf{x}(t) = C_1 e^{2t} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$
(B) $\mathbf{x}(t) = C_1 e^{2t} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix}$
(C) $\mathbf{x}(t) = C_1 e^{2t} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$
(D) $\mathbf{x}(t) = C_1 e^{2t} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$

3. (Fall 2016, Exam 2, Question 5) Consider a certain 2×2 linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a matrix of real numbers. Suppose all of its nonzero solutions are bounded, but they do not reach a limit as $t \to +\infty$. Which of the following is a possible pair of eigenvalues of $\mathbf{A}$?

(A) $-1, -4$
(B) $-1 \pm 4i$
(C) $\pm i$
(D) $-1, 4$

4. (Spring 2016, Exam 2, Question 6) Find all possible values of α such that the linear system below have an asymptotically stable improper node at $(0, 0)$?

$$\mathbf{x}' = \begin{bmatrix} -3\alpha & 5 \\ -5 & -2\alpha \end{bmatrix} \mathbf{x}.$$

(A) $\alpha = 10$
(B) $\alpha = -10$
(C) $\alpha = 10$ and $\alpha = -10$
(D) None of the above.

5. Determine whether each statement below is **TRUE or FALSE**. You must justify your answers.

   (a) (Spring 2017, Exam 2, Question 7(d)) The system of linear equations $\mathbf{x}' = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ has a critical point at $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, which is a saddle point.

   (b) (Fall 2016, Exam 2, Question 7(d)) Some, but not all, nonzero solutions of $\mathbf{x}' = \begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix} \mathbf{x}$ diverge to infinity as $t \to +\infty$.

   (c) (Fall 2016, Exam 2, Question 7(f)) The nonlinear system $\begin{cases} x' = y \cdot F(x, y) \\ y' = x \cdot F(x, y) \end{cases}$, where F and G have continuous partial derivatives everywhere, always has a critical point at $(0, 0)$.

   (d) (Fall 2015, Exam 2, Question 8(g)) If a 2×2 matrix $\mathbf{A}$ has two complex eigenvalues $-1 \pm \beta i$, where $\beta > 0$, then the origin as the equilibrium solution of the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is asymptotically stable regardless of what the value β is.

6. (Spring 2017, Exam 2, Question 8) Consider the system of linear differential equations listed below.

$$A. \qquad \mathbf{x}' = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \mathbf{x}$$

$$B. \qquad \mathbf{x}' = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix} \mathbf{x}$$

$$C. \qquad \mathbf{x}' = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \mathbf{x}$$

$$D. \qquad \mathbf{x}' = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \mathbf{x}$$

   Before you start, determine the eigenvalue(s) of each system above, and write then down in the space beside each system.

   For each of parts (a) through (f) below, write down the letter corresponding to the system on this list that has the indicated property. There is only one correct answer to each part. However, a system may be re-used for more than one part.

   (a) Which system has a saddle point at $(0, 0)$?
   (b) Which system has an improper node at $(0, 0)$?
   (c) Which system is (neutrally) stable?
   (d) Which system has all of its solutions converge to $(0, 0)$ as $t \to \infty$?

(e) Which system has all its nonzero solutions diverge to infinity as $t \to \infty$?

(f) Which system has all of its solutions bounded, both as t approach $-\infty$ and ∞?

7. (Fall 2014, Exam 2, Question 9) Determine the type and stability of the critical point at $(0,0)$ for each of the 2×2 linear systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ whose general solutions are given below. For the type, give the actual name. For the stability, use the letter **A** if the point is asymptotically stable, **U** if it is unstable, **S** if it is (neutrally) stable.

	Type	Stability
(a) $\mathbf{x}(t) = C_1 e^{-t} \begin{bmatrix} 2\cos\beta t \\ 5\sin\beta t \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} -5\sin\beta t \\ 2\cos\beta t \end{bmatrix}$	________	________
(b) $\mathbf{x}(t) = C_1 \begin{bmatrix} \cos\beta t \\ 3\sin\beta t \end{bmatrix} + C_2 \begin{bmatrix} 3\sin\beta t \\ -\cos\beta t \end{bmatrix}$	________	________
(c) $\mathbf{x}(t) = C_1 e^{-t/2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-t/2} \begin{bmatrix} 2t \\ 2t+5 \end{bmatrix}$	________	________
(d) $\mathbf{x}(t) = C_1 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$	________	________
(d) $\mathbf{x}(t) = C_1 e^{(3-\sqrt{6})t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{(3+\sqrt{6})t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$	________	________

(f) Which of the system (a) through (e) above, as $t \to \infty$, has nonzero solutions that converge to $(0,0)$, and has other nonzero solutions that move unbounded away from $(0,0)$?

8. (Spring 2017, Exam 2, Question 11)

(a) Find the general solution of the system of linear equations

$$\mathbf{x}' = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix} \mathbf{x}.$$

(b) Find the solution satisfying $\mathbf{x}(0) = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$.

(c) Classify the type and stability of the critical point at $(0,0)$.

9. (Spring 2017, Exam 2, Question 12) Consider the autonomous nonlinear system:

$$x' = x^2 - 2xy$$
$$y' = y + xy - 1$$

(a) The system has 3 critical points. One of the critical point is $(0,1)$. Verify that $(0,1)$ is indeed a critical point. Then find the other 2 critical points.

(b) Linearize the system about the point $(0,1)$. Classify the type and stability of this critical point by examining the linearized system. Be sure to clearly state the linearized system's coefficient matrix and its eigenvalues.