1 补充题 1

设 A 是线性空间 V 上的一个线性变换, 且 $A^2 = A$. 证明:

1. V 中任一向量 α 可分解为 $\alpha = \alpha_1 + \alpha_2$, 其中 $A\alpha_1 = \alpha_1, A\alpha_2 = 0$, 且这种分解是唯一的;
2. 若 $A\alpha = -\alpha$, 则 $\alpha = 0$.

证明:

1. $\forall \alpha \in V$, 定义 $\alpha_1 := A\alpha$, $\alpha_2 := \alpha - A\alpha$. 不难验证, $A\alpha_1 = A^2\alpha = A\alpha = \alpha_1$, 且 $A\alpha_2 = A\alpha - A^2\alpha = 0$. 存在性得证;

   对任意 α, 假设存在 α_1, α_2 和 α_1', α_2' 满足: $\alpha = \alpha_1 + \alpha_2 = \alpha_1' + \alpha_2'$ 以及 $A\alpha_1 = \alpha_1$, $A\alpha_1' = \alpha_1'$ 和 $A\alpha_2 = A\alpha_2' = 0$. 则有:

   $$\alpha_1 - \alpha_1' = A(\alpha_1 - \alpha_1') = -A(\alpha_2 - \alpha_2') = 0,$$

   从而 $\alpha_1 = \alpha_1'$, $\alpha_2 = \alpha_2'$. 唯一性得证.

2. 对 α, 存在满足题目条件的分解 $\alpha = \alpha_1 + \alpha_2$, 从而 $-\alpha = A\alpha = \alpha_1$, $\alpha_2 = 2\alpha$.

   $$\alpha = -A\alpha = -\frac{1}{2}A\alpha_2 = 0.$$

Q.E.D.