幂零阵 Jordan 标准形的证明：

$\boxed{\text{定理}}$ $\underset{\sim}{A}$ 为 V上幂零变换．则存在 $\alpha_1, \cdots, \alpha_s$ 及 $k_1, \cdots k_s$．使得

$$\underset{\sim}{A}^{k_i}\alpha_i = 0$$

且．

$$\alpha_1, \cdots, \underset{\sim}{A}^{k_1-1}\alpha_1, \cdots\cdots, \alpha_s, \cdots, \underset{\sim}{A}^{k_s-1}\alpha_s$$

构成 V中一组基．

证明：归纳法 $n=1$ √

假设小于n 情况正确．考虑．

$$Im(\underset{\sim}{A}) = \{\underset{\sim}{A}\alpha \mid \alpha \in V\}$$

由于 $\underset{\sim}{A}$ 幂零．$\Rightarrow$ $\underset{\sim}{A}$ 必有零特征向量 $\Rightarrow$ $\dim Im(\underset{\sim}{A}) < n$.

由于 $\underset{\sim}{A}\, Im(\underset{\sim}{A}) \subset Im(\underset{\sim}{A})$．且 $\dim Im(\underset{\sim}{A}) < n$．归纳假设知．

必有 $\beta_1, \cdots \beta_e$．及 $r_1, \cdots r_e$，使得．

$$\underset{\sim}{A}^{r_i}\beta_i = 0$$

且 $\beta_1, \cdots, \underset{\sim}{A}^{r_1-1}\beta_1, \cdots\cdots, \beta_e, \cdots, \underset{\sim}{A}^{r_e-1}\beta_e$ 构成 $Im(\underset{\sim}{A})$ 中一组基． --(*)

对于 ∀i $\beta_i \in Im(\underset{\sim}{A})$．于是有．$\delta_i \in V$．满足

$$\underset{\sim}{A}\delta_i = \beta_i.$$

并且可知 $\underset{\sim}{A}^{r_i}\delta_i = \underset{\sim}{A}^{r_i-1}\beta_i \in Ker(\underset{\sim}{A})$．由于 $\underset{\sim}{A}^{r_i-1}\beta_i$ 线性无关．現将其扩充为 $Ker(\underset{\sim}{A})$ 中一组基．

$$\underset{\sim}{A}^{r_1}\delta_1, \cdots, \underset{\sim}{A}^{r_e}\delta_e, \quad \theta_1, \cdots, \theta_m. \quad \text{--(**)}$$

下面证明：$\delta_1, \cdots, \underset{\sim}{A}^{r_1}\delta_1, \delta_2, \cdots \underset{\sim}{A}^{r_2}\delta_2, \cdots, \delta_e, \cdots \underset{\sim}{A}^{r_e}\delta_e, \theta_1 \cdots \theta_m$ --(***)

为V上的一组基．

1° 线性无关：若

$$a_{10}\delta_1 + a_{11}\underset{\sim}{A}\delta_1 + \cdots + a_{1r_1}\underset{\sim}{A}^{r_1}\delta_1$$
$$+ \cdots$$
$$+ a_{e0}\delta_e + a_{e1}\underset{\sim}{A}\delta_e + \cdots + a_{er_e}\underset{\sim}{A}^{r_e}\delta_e$$
$$+ b_1\theta_1 + \cdots + b_m\theta_m = 0$$

两边作用$\underset{\sim}{A}$ 得到

$$\underbrace{a_{10}\underset{\sim}{A}\delta_1}_{\beta_1} + a_{11}\underbrace{\underset{\sim}{A}^2\delta_1}_{\underset{\sim}{A}\beta_1} + \cdots + a_{1r_1}\underbrace{\underset{\sim}{A}^{r_1+1}\delta_1}_{\underset{\sim}{A}^{r_1}\beta_1 = 0} + \cdots + \overbrace{a_{e0}\underset{\sim}{A}\delta_1}^{\beta_e} + a_{e1}\underbrace{\underset{\sim}{A}^2\delta_e}_{\underset{\sim}{A}\beta_e} + \cdots + a_{er_e}\underbrace{\underset{\sim}{A}^{r_e+1}\delta_e}_{0} = 0$$

由(*)，故知 $a_{10} = a_{11} = \cdots = a_{1r_1-1} = 0$. $\cdots$ $a_{e0} = a_{e1} = \cdots = a_{er_e-1} = 0$.

再由(**) 知. $a_{ir_i} = 0$ ，$b_i = 0$

得线性无关.

2° 构成基.

$$\dim(\mathrm{Im}(\underset{\sim}{A})) = r_1 + \cdots + r_e$$

$$\dim(\mathrm{Ker}(\underset{\sim}{A})) = e + m.$$

故 $\dim V = \dim(\mathrm{Im}\,\underset{\sim}{A}) + \dim(\mathrm{Ker}\,\underset{\sim}{A})$
$= r_1 + \cdots + r_e + e + m$
$= (r_1+1) + \cdots + (r_e+1) + m.$ 为向量组(***)的个数.

故证明了 $\delta_1, \cdots, \underset{\sim}{A}^{r_1}\delta_1, \cdots, \delta_e, \cdots, \underset{\sim}{A}^{r_e}\delta_e, \theta_1, \cdots \theta_m$ 为 V上一组基.

得证明了 n 情况也正确. ▨