2024秋，有限元方法II，作业5

交作业时间：2024/12/20

1. Let $\hat{K} \subset \mathbb{R}^n$, F be a smooth mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$, and $K = F(\hat{K})$. Assume that F is globally invertible on K and its Jacobian DF is invertible. For any $\underset{\sim}{\hat{q}} \in (C^\infty(\hat{K}))^n$, define

$$\mathcal{G}(\underset{\sim}{\hat{q}})(x) := \frac{1}{J(\hat{x})} DF(\hat{x}) \underset{\sim}{\hat{q}}(\hat{x}), \quad \hat{x} = F^{-1}(x),$$

where $J(x) = |\det DF(\hat{x})|$. Show that

$$\mathrm{div} \underset{\sim}{q} = \frac{1}{J} \widehat{\mathrm{div}} \underset{\sim}{\hat{q}}.$$

Here, $\widehat{\mathrm{div}}$ means the derivatives on $\hat{x}$.

2. Let $v = \mathcal{F}(\hat{v}) := \hat{v}(F^{-1}(x))$, and $\underset{\sim}{q} = \mathcal{G}(\underset{\sim}{\hat{q}})$, where $F(\hat{x}) = B\hat{x} + b_0$ is an affine mapping. Show that

$$\begin{aligned}
\int_K \underset{\sim}{q} \cdot \mathrm{grad} v \, \mathrm{d}x &= \int_{\hat{K}} \underset{\sim}{\hat{q}} \cdot \widehat{\mathrm{grad}} \hat{v} \, \mathrm{d}\hat{x}, \\
\int_K v \, \mathrm{div} \underset{\sim}{q} \, \mathrm{d}x &= \int_{\hat{K}} \hat{v} \, \widehat{\mathrm{div}} \underset{\sim}{\hat{q}} \, \mathrm{d}\hat{x}, \\
\int_{\partial K} \underset{\sim}{q} \cdot \underset{\sim}{n} v \, \mathrm{d}s &= \int_{\partial \hat{K}} \underset{\sim}{\hat{q}} \cdot \underset{\sim}{\hat{n}} \hat{v} \, \mathrm{d}\hat{s}.
\end{aligned}$$

3. Given any $\epsilon > 0$, let $X = H(\mathrm{curl}) \cap \underset{\sim}{H}^{1/2+\epsilon}$. Given a Lipschitz domain Ω, show that there exists $\delta(\epsilon, \Omega) > 0$ so that $\mathrm{curl} X(\Omega) \subset \underset{\sim}{L}^{2+\delta(\epsilon,\Omega)}(\Omega)$. Let $W = H(\mathrm{div}) \cap \underset{\sim}{L}^{2+\delta(\epsilon)}$. Choose one of the following to prove the commutative diagram:

$$\begin{array}{ccc}
X(K) & \xrightarrow{\mathrm{curl}} & W(K) \\
\Pi_k^N \downarrow & & \downarrow \Pi_k^{RT} \\
N_k(K) & \xrightarrow{\mathrm{curl}} & RT_k(K)
\end{array}
\qquad
\begin{array}{ccc}
X(K) & \xrightarrow{\mathrm{curl}} & W(K) \\
\Pi_{k+1}^{NC} \downarrow & & \downarrow \Pi_k^{RT} \\
NC_{k+1}(K) & \xrightarrow{\mathrm{curl}} & RT_k(K)
\end{array}$$

Find the similar version for $BDM_k(K)$ (no need to show the proof).

4. Show that the following two inequalities are equivalent

$$\|p\|_{L^2(\Omega)} \lesssim \|p\|_{H^{-1}(\Omega)} + \sum_{i=1}^{d} \|\frac{\partial p}{\partial x_i}\|_{H^{-1}(\Omega)} \quad \forall p \in L^2(\Omega), \tag{1}$$

$$\|p\|_{L^2(\Omega)} \lesssim \sum_{i=1}^{d} \|\frac{\partial p}{\partial x_i}\|_{H^{-1}(\Omega)} \quad \forall p \in L_0^2(\Omega). \tag{2}$$

5. Let Ω be a connected domain with a Lipschitz boundary. Assume that $\Gamma_D \subset \partial\Omega$ satisfies $\mathrm{meas}(\Gamma_D) \neq 0$. Show that

$$\|\underset{\sim}{v}\|_{\underset{\sim}{H}^1(\Omega)} \lesssim \|\boldsymbol{\varepsilon}(\underset{\sim}{v})\|, \quad \forall \underset{\sim}{v} \in \underset{\sim}{H}_D^1(\Omega),$$

where $\underset{\sim}{H}_D^1(\Omega) := \{\underset{\sim}{v} \in \underset{\sim}{H}^1(\Omega) : \ \underset{\sim}{v} = 0 \text{ on } \Gamma_D\}$.

6. For the Stokes pair $\underset{\sim}{\mathcal{P}}_1^{\mathrm{CR}}$-$\mathcal{P}_0^{-1}$, prove that it satisfies the following discrete inf-sup condition:

$$\inf_{q_h \in Q_h} \sup_{\underset{\sim}{v}_h \in V_h} \frac{(\mathrm{div}_h \underset{\sim}{v}_h, q_h)}{\|\underset{\sim}{v}_h\|_{1,h} \|q_h\|_{L^2}} \gtrsim 1,$$

where div_h denotes the piecewise divergence, and $\|\cdot\|_{1,h}$ represents the piecewise H^1-norm. For the aforementioned element, provide the error estimate for the following numerical scheme ($\nu = 1$):

$$\begin{cases} 2(\boldsymbol{\varepsilon}_h(\underset{\sim}{u}_h), \boldsymbol{\varepsilon}_h(\underset{\sim}{v}_h)) - (\mathrm{div}_h \underset{\sim}{v}_h, p_h) = (\underset{\sim}{f}, \underset{\sim}{v}_h) & \forall \underset{\sim}{v}_h \in V_h, \\ \qquad\qquad -(\mathrm{div}_h \underset{\sim}{u}_h, q_h) = 0 & \forall q_h \in Q_h. \end{cases}$$

7. Let $V = \underset{\sim}{H}_0^1(\Omega)$. Consider $V^0 = \{\underset{\sim}{v} \in V \mid \int_T \mathrm{div}\underset{\sim}{v}\, \mathrm{d}x = 0, \forall T \in \mathcal{T}_h\}$, define $\Pi_2 : V^0 \to B(\mathrm{gradQ_h})$ by

$$\begin{cases} \Pi_2 \underset{\sim}{v}|_T \in B(\mathrm{gradQ_h})|_T, \\ \displaystyle\int_T \mathrm{div}(\Pi_2 \underset{\sim}{v} - \underset{\sim}{v})\, \mathrm{d}x = 0, \quad \forall q_h \in Q_h|_T, \end{cases} \quad \forall T \in \mathcal{T}_h.$$

Show that $\|\Pi_2 \underset{\sim}{v}\|_1 \lesssim \|\underset{\sim}{v}\|_1$ for $\underset{\sim}{v} \in V^0$.

8. Consider the Stokes problem with homogeneous Dirichlet boundary condition:

$$-\Delta \underset{\sim}{u} + \nabla p = \underset{\sim}{f} \quad \text{in } \Omega,$$
$$-\mathrm{div}\underset{\sim}{u} = 0 \quad \text{in } \Omega,$$
$$\underset{\sim}{u}|_{\partial\Omega} = \underset{\sim}{0}.$$

Let $V = \underset{\sim}{H}_0^1(\Omega)$ and $Q = L_0^2(\Omega)$. Given a stable Stokes pair $V_h \times Q_h \subset V \times Q$, we can obtain the following energy energy estimate

$$\|\underset{\sim}{u} - \underset{\sim}{u}_h\|_{H^1} + \|p - p_h\|_{L^2} \lesssim \inf_{\underset{\sim}{v}_h \in V_h} \|\underset{\sim}{u} - \underset{\sim}{v}_h\|_{H^1} + \inf_{q_h \in Q_h} \|p - q_h\|_{L^2}.$$

Assume further the approximation property of $V_h \times Q_h$:

$$\inf_{\underset{\sim}{v}_h \in V_h} \|\underset{\sim}{z} - \underset{\sim}{v}_h\|_{H^1} \lesssim h\|\underset{\sim}{z}\|_{H^2} \quad \forall \underset{\sim}{z} \in \underset{\sim}{H}^2(\Omega),$$

$$\inf_{q_h \in Q_h} \|r - q_h\|_{L^2} \lesssim h\|r\|_{H^1}, \quad \forall r \in H^1(\Omega).$$

Duality argument: Find appropriate regularity assumption of the dual problem:

$$\begin{aligned} -\Delta \underset{\sim}{z} + \nabla r &= \underset{\sim}{\theta} \quad \text{in } \Omega, \\ \operatorname{div} \underset{\sim}{z} &= 0 \quad \text{in } \Omega, \\ \underset{\sim}{z}|_{\partial\Omega} &= \underset{\sim}{0}. \end{aligned}$$

so that one can obtain the L^2 estimate of $\underset{\sim}{u}$:

$$\|\underset{\sim}{u} - \underset{\sim}{u}_h\|_{L^2} \lesssim h \left(\inf_{\underset{\sim}{v}_h \in V_h} \|\underset{\sim}{u} - \underset{\sim}{v}_h\|_{H^1} + \inf_{q_h \in Q_h} \|p - q_h\|_{L^2} \right).$$