2024秋，有限元方法II，作业4

交作业时间：2024/11/28

1. Given a triangle T with diameter h_T, if $r \in \mathbb{R}$, show that

$$h_T \|r\|_{L^2(T)} \lesssim \|r\|_{H^{-1}(T)},$$

where the hidden constant depends on the shape-regularity of T.

2. Let V, Q be Banach spaces, $B : V \to Q'$ be a bound linear operator, $Z := N(B)$. For any subspace $S \subset V$, define

$$S^\circ := \{f \in V' \mid \langle f, v\rangle = 0,\ \forall v \in S\}.$$

For any subspace $F \subset V'$, define

$${}^\circ F := \{v \in V \mid \langle f, v\rangle = 0,\ \forall f \in F\}.$$

- Show that S° and ${}^\circ F$ are closed.
- Show that ${}^\circ(S^\circ) = S$ if and only if S is closed in V; And $({}^\circ F)^\circ = F$ if and only if F is closed in V'.
- Show that ${}^\circ R(B') = Z$.
- Show that $R(B') = Z^\circ$ if and only if $R(B')$ is closed in V'.

3. Let H be a Hilbert space with a norm $\|\cdot\|_H$ and inner product $(\cdot,\cdot)_H$. Let $P : H \to H$ be an idempotent, such that $0 \neq P^2 = P \neq I$. Then, the following indentity holds

$$\|P\|_{\mathcal{L}(H,H)} = \|I - P\|_{\mathcal{L}(H,H)}.$$

4. Define

$$\mathbb{H}_k(K) := \{\underset{\sim}{q} \in \underset{\sim}{\mathcal{P}}_k(K) \mid \operatorname{div}\underset{\sim}{q} = 0 \quad \text{and} \quad \underset{\sim}{q} \cdot \underset{\sim}{n}|_{\partial K} = 0\}.$$

Show that in 2D, $\mathbb{H}_k(K) = \operatorname{curl}(b_K \mathcal{P}_{k-2}(K))$. (Hint: You can directly utilize the dimension formula for $\mathbb{H}_k(K)$ discussed in class.)

5. Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain. For any $\underset{\sim}{v} \in (L^2(\Omega))^2$, show that there exists $\psi \in H_0^1(\Omega)$ and $\phi \in H^1(\Omega)$, such that

$$\underset{\sim}{v} = \nabla\psi + \operatorname{curl}\phi,$$

and

$$\|\nabla\phi\|_{L^2} + \|\nabla\psi\|_{L^2} \lesssim \|\underset{\sim}{v}\|_{L^2}.$$

(Hint: $R(\operatorname{curl}) = N(\operatorname{div})$ on simply connected domain.)

6. Let $\Omega \subset \mathbb{R}^3$ be the half space $x_3 < 0$ and Γ be the space $x_3 = 0$. Given a vector $\underset{\sim}{\chi} = (\chi_1, \chi_2, \chi_3)^T \in H(\mathrm{curl}, \Omega)$, show that $\mathrm{div}_\Gamma \mathrm{Tr}\underset{\sim}{\chi} = \mathrm{curl}\underset{\sim}{\chi} \cdot \underset{\sim}{n}|_\Gamma$, where $\mathrm{Tr}\chi = \underset{\sim}{\chi} \times \underset{\sim}{n}$.

7. For any $\underset{\sim}{w} \in \underset{\sim}{\mathcal{H}}_k$ ($\mathcal{H}_k$ stands for the homogeneous polynomial space of order k), define
$$\underset{\sim}{\eta} = \frac{\mathrm{curl}\underset{\sim}{w}}{k+1}, \quad \mu = \frac{\underset{\sim}{x} \cdot \underset{\sim}{w}}{k+1}.$$
show that
$$-\underset{\sim}{x} \times \underset{\sim}{\eta} + \nabla \mu = \underset{\sim}{w}.$$

8. Show that for any $\underset{\sim}{p}_k \in \underset{\sim}{\mathcal{P}}_k$, there exists a decomposition
$$\underset{\sim}{p}_k = \underset{\sim}{w}_{k-1} + \nabla \theta,$$
where $\underset{\sim}{w}_{k-1} \in \underset{\sim}{\mathcal{P}}_{k-1} + \underset{\sim}{x} \times \underset{\sim}{\mathcal{P}}_{k-1}$, and $\theta \in \mathcal{P}_{k+1}$.