2023秋，有限元方法II，作业5

交作业时间：2023/12/19

1. Given any $\epsilon > 0$, let $X = H(\text{curl}) \cap \underset{\sim}{H}^{1/2+\epsilon}$. Given a Lipschitz domain Ω, show that there exists $\delta(\epsilon, \Omega) > 0$ so that $\text{curl}X(\Omega) \subset \underset{\sim}{L}^{2+\delta(\epsilon,\Omega)}(\Omega)$. Let $W = H(\text{div}) \cap \underset{\sim}{L}^{2+\delta(\epsilon)}$. Show the following commutative diagram

$$\begin{array}{ccc} X(K) & \xrightarrow{\text{curl}} & W(K) \\ \Pi_k^N \big\downarrow & & \big\downarrow \Pi_k^{RT} \\ N_k(K) & \xrightarrow{\text{curl}} & RT_k(K) \end{array} \qquad \begin{array}{ccc} X(K) & \xrightarrow{\text{curl}} & W(K) \\ \Pi_{k+1}^{NC} \big\downarrow & & \big\downarrow \Pi_k^{RT} \\ NC_{k+1}(K) & \xrightarrow{\text{curl}} & RT_k(K) \end{array}$$

Find the similar version for $BDM_k(K)$ (no need to show the proof).

2. Show that the following two inequalities are equivalent

$$\|p\|_{L^2(\Omega)} \lesssim \|p\|_{H^{-1}(\Omega)} + \sum_{i=1}^{d} \|\frac{\partial p}{\partial x_i}\|_{H^{-1}(\Omega)} \quad \forall p \in L^2(\Omega), \tag{1}$$

$$\|p\|_{L^2(\Omega)} \lesssim \sum_{i=1}^{d} \|\frac{\partial p}{\partial x_i}\|_{H^{-1}(\Omega)} \quad \forall p \in L_0^2(\Omega). \tag{2}$$

3. Let $\mathcal{T}_h$ is a shape-regular triangulation of $\Omega \subset \mathbb{R}^2$. For any edge e connected by V_i and V_j, define

$$b_e := 6\phi_i(x)\phi_j(x)/|e|,$$

where $\phi_i(x)$ is the piecewise linear basis function associated with V_i. Let

$$\Pi_h \underset{\sim}{v} := \sum_e \left(\int_e \underset{\sim}{v} \text{d}s \right) b_e.$$

Show that

$$\|\Pi_h \underset{\sim}{v}\|_{L^2}^2 \lesssim \|\underset{\sim}{v}\|_{L^2}^2 + h^2 |\underset{\sim}{v}|_{H^1}^2.$$

4. For the Stokes problem, assume the modified inf-sup condition

$$\inf_{q_h \in Q_h} \sup_{\underset{\sim}{v}_h \in V_h} \frac{(\text{div}\underset{\sim}{v}_h, q_h)}{\|\underset{\sim}{v}_h\|_V \|q_h - \bar{q}_h\|_Q} = k_0 > 0,$$

where k_0 is independent of h and $\bar{q}_h$ is the L^2 projection of q_h. Assume further that V_h is such that, for any $q_h \in \mathcal{P}_0^{-1}$,

$$\sup_{\underset{\sim}{v}_h \in V_h} \frac{(\mathrm{div}\underset{\sim}{v}_h, q_h)}{\|\underset{\sim}{v}_h\|_V} \geq \gamma_0 \|q_h\|_Q,$$

with γ_0 independent of h. Then, $V_h \times Q_h$ is inf-sup stable.

5. Consider the Stokes problem with homogeneous Dirichlet boundary condition:

$$\begin{aligned} -\Delta \underset{\sim}{u} + \nabla p &= \underset{\sim}{f} \quad \text{in } \Omega, \\ -\mathrm{div}\underset{\sim}{u} &= 0 \quad \text{in } \Omega, \\ \underset{\sim}{u}|_{\partial\Omega} &= \underset{\sim}{0}. \end{aligned}$$

Let $V = \underset{\sim}{H}_0^1(\Omega)$ and $Q = L_0^2(\Omega)$. Given a stable Stokes pair $V_h \times Q_h \subset V \times Q$, we can obtain the following energy energy estimate

$$\|\underset{\sim}{u} - \underset{\sim}{u}_h\|_{H^1} + \|p - p_h\|_{L^2} \lesssim \inf_{\underset{\sim}{v}_h \in V_h} \|\underset{\sim}{u} - \underset{\sim}{v}_h\|_{H^1} + \inf_{q_h \in Q_h} \|p - q_h\|_{L^2}.$$

Assume further the approximation property of $V_h \times Q_h$:

$$\begin{aligned} \inf_{\underset{\sim}{v}_h \in V_h} \|\underset{\sim}{z} - \underset{\sim}{v}_h\|_{H^1} &\lesssim h\|\underset{\sim}{z}\|_{H^2} \quad \forall \underset{\sim}{z} \in \underset{\sim}{H}^2(\Omega), \\ \inf_{q_h \in Q_h} \|r - q_h\|_{L^2} &\lesssim h\|r\|_{H^1}, \quad \forall r \in H^1(\Omega). \end{aligned}$$

Duality argument: Find appropriate regularity assumption of the dual problem:

$$\begin{aligned} -\Delta \underset{\sim}{z} + \nabla r &= \underset{\sim}{\theta} \quad \text{in } \Omega, \\ \mathrm{div}\underset{\sim}{z} &= 0 \quad \text{in } \Omega, \\ \underset{\sim}{z}|_{\partial\Omega} &= \underset{\sim}{0}. \end{aligned}$$

so that one can obtain the L^2 estimate of $\underset{\sim}{u}$:

$$\|\underset{\sim}{u} - \underset{\sim}{u}_h\|_{L^2} \lesssim h \left(\inf_{\underset{\sim}{v}_h \in V_h} \|\underset{\sim}{u} - \underset{\sim}{v}_h\|_{H^1} + \inf_{q_h \in Q_h} \|p - q_h\|_{L^2} \right).$$