# 2023秋，有限元方法II，作业4

交作业时间：2023/12/7

1. Let $V, Q$ be Banach spaces, $B : V \to Q'$ be a bound linear operator, $Z := N(B)$. For any $S \subset V$, define
$$S^\circ := \{f \in V' \mid \langle f, v\rangle = 0,\ \forall v \in S\}.$$
For any $F \subset V'$, define
$$^\circ F := \{v \in V \mid \langle f, v\rangle = 0,\ \forall f \in F\}.$$
   - Show that $S^\circ$ and $^\circ F$ are closed.
   - Show that $^\circ(S^\circ) = S$ if and only if $S$ is closed in $V$; And $(^\circ F)^\circ = F$ if and only if $F$ is closed in $V'$.
   - Show that $^\circ R(B') = Z$.
   - Show that $R(B') = Z^\circ$ if and only if $R(B')$ is closed in $V'$.

2. Let $H$ be a Hilbert space with a norm $\|\cdot\|_H$ and inner product $(\cdot,\cdot)_H$. Let $P : H \to H$ be an idempotent, such that $0 \neq P^2 = P \neq I$. Then, the following indentity holds
$$\|P\|_{\mathcal{L}(H,H)} = \|I - P\|_{\mathcal{L}(H,H)}.$$

3. Let $\hat{K} \subset \mathbb{R}^n$, $F$ be a smooth mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$, and $K = F(\hat{K})$. Assume that $F$ is globally invertible on $K$ and its Jacobian $DF$ is invertible. For any $\underset{\sim}{\hat{q}} \in (C^\infty(\hat{K}))^n$, define
$$\mathcal{G}(\underset{\sim}{\hat{q}})(x) := \frac{1}{J(\hat{x})} DF(\hat{x})\underset{\sim}{\hat{q}}(\hat{x}), \quad \hat{x} = F^{-1}(x),$$
where $J(x) = |\det DF(\hat{x})|$. Show that
$$\mathrm{div}\underset{\sim}{q} = \frac{1}{J}\widehat{\mathrm{div}}\underset{\sim}{\hat{q}}.$$
Here, $\widehat{\mathrm{div}}$ means the derivatives on $\hat{x}$.

4. Define
$$\mathbb{H}_k(K) := \{\underset{\sim}{q} \in \underset{\sim}{\mathcal{P}}_k(K) \mid \mathrm{div}\underset{\sim}{q} = 0 \quad \text{and} \quad \underset{\sim}{q} \cdot \underset{\sim}{n}|_{\partial K} = 0\}.$$
Show that in 2D, $\mathbb{H}_k(K) = \mathrm{curl}(b_K \mathcal{P}_{k-2}(K))$. Count the dimension of $\mathbb{H}_k(K)$ in 2D.

5. Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain. For any $\underset{\sim}{v} \in (L^2(\Omega))^2$, show that there exists $\psi \in H_0^1(\Omega)$ and $\phi \in H^1(\Omega)$, such that

$$\underset{\sim}{v} = \nabla\psi + \mathrm{curl}\phi,$$

and

$$\|\nabla\phi\|_{L^2} + \|\nabla\psi\|_{L^2} \lesssim \|\underset{\sim}{v}\|_{L^2}.$$

(Hint: $R(\mathrm{curl}) = N(\mathrm{div})$ on simply connected domain.)

6. For any $\underset{\sim}{w} \in \underset{\sim}{\mathcal{H}}_k$ ($\mathcal{H}_k$ stands for the homogeneous polynomial space of order $k$), define

$$\underset{\sim}{\eta} = \frac{\mathrm{curl}\underset{\sim}{w}}{k+1}, \quad \mu = \frac{\underset{\sim}{x} \cdot \underset{\sim}{w}}{k+1}.$$

show that

$$\underset{\sim}{x} \times \underset{\sim}{\eta} + \nabla\mu = \underset{\sim}{w}.$$

7. Show that for any $\underset{\sim}{p}_k \in \underset{\sim}{\mathcal{P}}_k$, there exists a decomposition

$$\underset{\sim}{p}_k = \underset{\sim}{w}_{k-1} + \nabla\theta,$$

where $\underset{\sim}{w}_{k-1} \in \underset{\sim}{\mathcal{P}}_{k-1} + \underset{\sim}{x} \times \underset{\sim}{\mathcal{P}}_{k-1}$, and $\theta \in \mathcal{P}_{k+1}$.