2022年秋，有限元方法II，上机作业2

截至时间：2022/12/25，晚上12点

要求:

- 用TeX写上机报告(中英文均可)，包含必要的数值结果讨论，页数上限15.
- 本次上机作业中，须自己组装刚度矩阵，推荐使用软件包iFEM.
- 截止时间前将程序和上机报告的源码发送至snwu@math.pku.edu.cn

Consider the following mixed formulation of the Poisson equation

$$\begin{cases} \boldsymbol{p} - \nabla u = 0 & \text{in } \Omega \subset \mathbb{R}^2, \\ -\mathrm{div}\, \boldsymbol{p} = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{1}$$

The computational domain is given as,

$$\Omega := \{(x, y) \in (-1, 1)^2 : 0 < \theta < \pi/\beta\},$$

where $\beta \geq \frac{1}{2}$. Note that if $\beta < 1$, then Ω is non-convex. Use $\mathrm{RT}_k - \mathcal{P}_k^-$ and $\mathrm{BDM}_{k+1} - \mathcal{P}_k^-$ mixed finite elements to solve (1) with different β's and exact solutions on *quasi-uniform* meshes. The source f and boundary data g can be obtained from the exact solution u.

Problem 1. Choose *smooth* solution u. Report the errors of $\boldsymbol{p}$ in $H(\mathrm{div})$ norm, L^2 norm and errors of u in L^2 norm for different β's.

Problem 2. Quasi-uniform meshes: Choose

$$u = r^\beta \sin(\beta\theta). \tag{2}$$

Report the errors and rates of $\boldsymbol{p}$ in $H(\mathrm{div})$, $L^2_{\alpha-1}$ norms and u in L^2, L^2_α norms with different $\alpha \in (1-\beta, 1+\beta)$ for different β's such that $\beta \in (\frac{1}{2}, 1]$ (non-convex case). Here, L^2_α denotes the weighted Sobolev space equipped with the norm

$$\|p\|_{L^2_\alpha} = \|r^\alpha p\|_{L^2(\Omega)},$$

where $r = \sqrt{x^2 + y^2}$ is the Euclidean distance to the origin. Try to summarize your findings.

<u>Remark:</u> The case in which $k = 0$ (RT_0-$\mathcal{P}_0^{-1}$ and BDM_1-$\mathcal{P}_0^{-1}$) is required. At least one high-order case (e.g., RT_1-$\mathcal{P}_1^{-1}$ or BDM_2-$\mathcal{P}_1^{-1}$) is required.