2022秋，有限元方法II，作业4

交作业时间：2022/11/18

1. Let V, Q be Banach spaces, $B : V \to Q'$ be a bound linear operator, $Z := N(B)$. For any $S \subset V$, define

$$S^\circ := \{f \in V' \mid \langle f, v \rangle = 0,\ \forall v \in S\}.$$

For any $F \subset V'$, define

$${}^\circ F := \{v \in V \mid \langle f, v \rangle = 0,\ \forall f \in F\}.$$

- Show that S° and ${}^\circ F$ are closed.
- Show that ${}^\circ(S^\circ) = S$ if and only if S is closed in V; And $({}^\circ F)^\circ = F$ if and only if F is closed in V'.
- Show that ${}^\circ R(B') = Z$.
- Show that $R(B') = Z^\circ$ if and only if $R(B')$ is closed in V'.

2. Let H be a Hilbert space with a norm $\|\cdot\|_H$ and inner product $(\cdot,\cdot)_H$. Let $P : H \to H$ be an idempotent, such that $0 \neq P^2 = P \neq I$. Then, the following identity holds

$$\|P\|_{\mathcal{L}(H,H)} = \|I - P\|_{\mathcal{L}(H,H)}.$$

3. Show that the following two inequalities are equivalent

$$\|p\|_{L^2(\Omega)} \lesssim \|p\|_{H^{-1}(\Omega)} + \sum_{i=1}^{d} \|\frac{\partial p}{\partial x_i}\|_{H^{-1}(\Omega)} \quad \forall p \in L^2(\Omega), \tag{1}$$

$$\|p\|_{L^2(\Omega)} \lesssim \sum_{i=1}^{d} \|\frac{\partial p}{\partial x_i}\|_{H^{-1}(\Omega)} \quad \forall p \in L_0^2(\Omega). \tag{2}$$

4. Let Ω be a connected domain with a Lipschitz boundary. Assume that $\Gamma_D \subset \partial\Omega$ satisfies $\mathrm{meas}(\Gamma_D) \neq 0$. Show that

$$\|\underset{\sim}{v}\|_{\underset{\sim}{H}^1(\Omega)} \lesssim \|\boldsymbol{\varepsilon}(\underset{\sim}{v})\|, \quad \forall \underset{\sim}{v} \in \underset{\sim}{H}_D^1(\Omega),$$

where $\underset{\sim}{H}_D^1(\Omega) := \{\underset{\sim}{v} \in \underset{\sim}{H}^1(\Omega) :\ \underset{\sim}{v} = 0 \text{ on } \Gamma_D\}$.

5. Let $\mathcal{T}_h$ is a shape-regular triangulation of $\Omega \subset \mathbb{R}^2$. For any edge e connected by V_i and V_j, define

$$b_e := 6\phi_i(x)\phi_j(x)/|e|,$$

where $\phi_i(x)$ is the piecewise linear basis function associated with V_i. Let

$$\Pi_h \underset{\sim}{v} := \sum_e \left(\int_e \underset{\sim}{v} \mathrm{d}s \right) b_e.$$

Show that

$$\|\Pi_h \underset{\sim}{v}\|_{L^2}^2 \lesssim \|\underset{\sim}{v}\|_{L^2}^2 + h^2 |\underset{\sim}{v}|_{H^1}^2.$$