2021秋，有限元方法II，作业4

交作业时间：2021/12/17

1. Let H be a Hilbert space with a norm $\|\cdot\|_H$ and inner product $(\cdot,\cdot)_H$. Let $P : H \to H$ be an idempotent, such that $0 \neq P^2 = P \neq I$. Then, the following indentity holds
$$\|P\|_{\mathcal{L}(H,H)} = \|I - P\|_{\mathcal{L}(H,H)}.$$

2. Let V, Q be Banach spaces, $B : V \to Q'$ be a bound linear operator, $Z := N(B)$. For any $S \subset V$, define
$$S^\circ := \{f \in V' \mid \langle f, v\rangle = 0,\ \forall v \in S\}.$$
For any $F \subset V'$, define
$$^\circ F := \{v \in V \mid \langle f, v\rangle = 0,\ \forall f \in F\}.$$
   - Show that S° and $^\circ F$ are closed.
   - Show that $^\circ(S^\circ) = S$ if and only if S is closed in V; And $(^\circ F)^\circ = F$ if and only if F is closed in V'.
   - Show that $^\circ R(B') = Z$.
   - Show that $R(B') = Z^\circ$ if and only if $R(B')$ is closed in V'.

3. Consider the Stokes problem with homogeneous Dirichlet boundary condition:
$$-\Delta \underset{\sim}{u} + \nabla p = \underset{\sim}{f} \quad \text{in } \Omega,$$
$$-\text{div}\underset{\sim}{u} = 0 \quad \text{in } \Omega,$$
$$\underset{\sim}{u}|_{\partial\Omega} = \underset{\sim}{0}.$$
Let $V = \underset{\sim}{H}_0^1(\Omega)$ and $Q = L_0^2(\Omega)$. Given a stable Stokes pair $V_h \times Q_h \subset V \times Q$, we can obtain the following energy energy estimate
$$\|\underset{\sim}{u} - \underset{\sim}{u}_h\|_{H^1} + \|p - p_h\|_{L^2} \lesssim \inf_{\underset{\sim}{v}_h \in V_h} \|\underset{\sim}{u} - \underset{\sim}{v}_h\|_{H^1} + \inf_{q_h \in Q_h} \|p - q_h\|_{L^2}.$$
Assume further the approximation property of $V_h \times Q_h$:
$$\inf_{\underset{\sim}{v}_h \in V_h} \|\underset{\sim}{z} - \underset{\sim}{v}_h\|_{H^1} \lesssim h\|\underset{\sim}{z}\|_{H^2} \quad \forall \underset{\sim}{z} \in \underset{\sim}{H}^2(\Omega),$$
$$\inf_{q_h \in Q_h} \|r - q_h\|_{L^2} \lesssim h\|r\|_{H^1}, \quad \forall r \in H^1(\Omega).$$

<u>Duality argument</u>: Find appropriate regularity assumption of the dual problem:

$$\begin{aligned} -\Delta \underset{\sim}{z} + \nabla r &= \underset{\sim}{\theta} \quad \text{in } \Omega, \\ \operatorname{div} \underset{\sim}{z} &= 0 \quad \text{in } \Omega, \\ \underset{\sim}{z}|_{\partial\Omega} &= \underset{\sim}{0}. \end{aligned}$$

so that one can obtain the L^2 estimate of $\underset{\sim}{u}$:

$$\|\underset{\sim}{u} - \underset{\sim}{u}_h\|_{L^2} \lesssim h \left(\inf_{\underset{\sim}{v}_h \in V_h} \|\underset{\sim}{u} - \underset{\sim}{v}_h\|_{H^1} + \inf_{q_h \in Q_h} \|p - q_h\|_{L^2} \right).$$

4. Let $\mathcal{T}_h$ is a shape-regular triangulation of $\Omega \subset \mathbb{R}^2$. For any edge e connected by V_i and V_j, define

$$b_e := 6\phi_i(x)\phi_j(x)/|e|,$$

where $\phi_i(x)$ is the piecewise linear basis function associated with V_i. Let

$$\Pi_h \underset{\sim}{v} := \sum_e \left(\int_e \underset{\sim}{v} \mathrm{d}s \right) b_e.$$

Show that

$$\|\Pi_h \underset{\sim}{v}\|_{L^2}^2 \lesssim \|\underset{\sim}{v}\|_{L^2}^2 + h^2 |\underset{\sim}{v}|_{H^1}^2.$$

5. Let $\hat{K} \subset \mathbb{R}^n$, F be a smooth mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$, and $K = F(\hat{K})$. Assume that F is globally invertible on K and its Jacobian DF is invertible. For any $\underset{\sim}{\hat{q}} \in (C^\infty(\hat{K}))^n$, define

$$\mathcal{G}(\underset{\sim}{\hat{q}})(x) := \frac{1}{J(\hat{x})} DF(\hat{x}) \underset{\sim}{\hat{q}}(\hat{x}), \quad \hat{x} = F^{-1}(x),$$

where $J(x) = |\det DF(\hat{x})|$. Show that

$$\operatorname{div} \underset{\sim}{q} = \frac{1}{J} \widehat{\operatorname{div}} \underset{\sim}{\hat{q}}.$$

Here, $\widehat{\operatorname{div}}$ means the derivatives on $\hat{x}$.

6. Let $v = \mathcal{F}(\hat{v}) := \hat{v}(F^{-1}(x))$, and $\underset{\sim}{q} = \mathcal{G}(\underset{\sim}{\hat{q}})$, where $F(\hat{x}) = B\hat{x} + b_0$ is an affine mapping. Show that

$$\begin{aligned} \int_K \underset{\sim}{q} \cdot \operatorname{grad} v \, \mathrm{d}x &= \int_{\hat{K}} \underset{\sim}{\hat{q}} \cdot \widehat{\operatorname{grad}} \hat{v} \, \mathrm{d}\hat{x}, \\ \int_K v \operatorname{div} \underset{\sim}{q} \, \mathrm{d}x &= \int_{\hat{K}} \hat{v} \, \widehat{\operatorname{div}} \underset{\sim}{\hat{q}} \, \mathrm{d}\hat{x}, \\ \int_{\partial K} \underset{\sim}{q} \cdot \underset{\sim}{n} v \, \mathrm{d}s &= \int_{\partial \hat{K}} \underset{\sim}{\hat{q}} \cdot \underset{\sim}{\hat{n}} \hat{v} \, \mathrm{d}\hat{s}. \end{aligned}$$

7. Define

$$\mathbb{H}_k(K) := \{\underset{\sim}{q} \in \underset{\sim}{\mathcal{P}}_k(K) \mid \text{div}\underset{\sim}{q} = 0 \quad \text{and} \quad \underset{\sim}{q} \cdot \underset{\sim}{n}|_{\partial K} = 0\}.$$

Show that in 2D, $\mathbb{H}_k(K) = \text{curl}(b_K \mathcal{P}_{k-2}(K))$.

8. Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain. For any $\underset{\sim}{v} \in (L^2(\Omega))^2$, show that there exists $\psi \in H_0^1(\Omega)$ and $\phi \in H^1(\Omega)$, such that

$$\underset{\sim}{v} = \nabla\psi + \text{curl}\phi,$$

and

$$\|\nabla\phi\|_{L^2} + \|\nabla\psi\|_{L^2} \lesssim \|\underset{\sim}{v}\|_{L^2}.$$

(Hint: $R(\text{curl}) = N(\text{div})$ on simply connected domain.)