2021秋，有限元方法II，作业3

交作业时间：2021/12/01

The Mathematical Theory of Finite Element Methods:

- Chapter 8: 8.x.13, 8.x.19, 8.x.20
- Chapter 9: 9.x.5, 9.x.7, 9.x.19
- Chapter 10: 10.x.4, 10.x.10, 10.x.11

Supplementary Questions:

1. Consider two different triangulations on the unit square $\Omega = (0,1)^2$: the uniform grid (Fig. 1) and the criss-cross grid (Fig. 2).

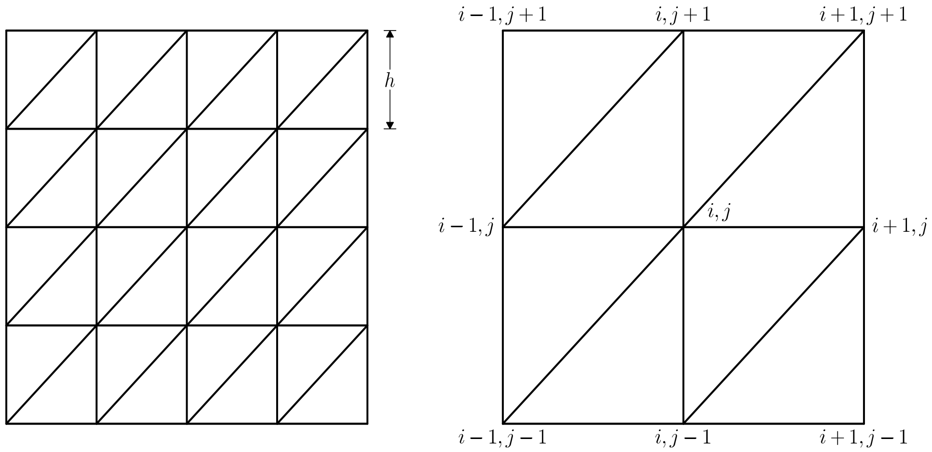

Figure 1: *Uniform triangulation of the domain* $(0,1)^2$

(a) For the uniform grid, let $u_h = \sum_{ij} u_{i,j}\phi_{i,j}$, where $\phi_{i,j}$ is the nodal basis function associated with the grid point (x_i, y_i). Show that the finite element formulation $a(u_h, \phi_{ij}) = (f, \phi_{ij})$ can be written as

$$\frac{-u_{i,j-1} - u_{i,j+1} + 4u_{i,j} - u_{i-1,j} - u_{i+1,j}}{h^2} = \frac{1}{h^2}\tilde{f}_{i,j}. \qquad (1)$$

Here

$$\tilde{f}_{i,j} = \frac{1}{h^2}\int_\Omega f(x,y)\phi_{ij}(x,y).$$

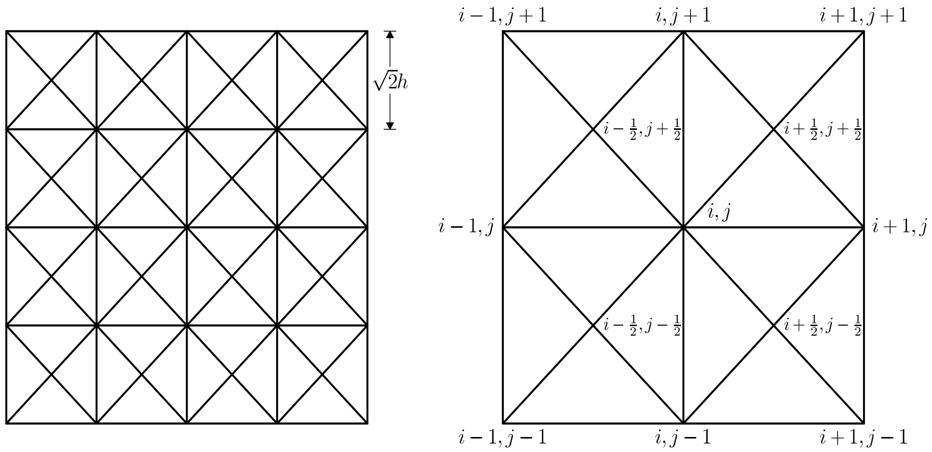

Figure 2: *Criss-cross triangulation of the domain* $(0,1)^2$

(b) Prove that $\tilde{f}_{i,j} - f(x_i, y_i) = \mathcal{O}(h^2)$ for the uniform grid.

(c) Show that on the criss-cross grid,

$$\frac{(\nabla u_h, \nabla \phi_{i,j})}{h^2} + \Delta u(x_i, y_i) = \mathcal{O}(h^2).$$

(d) Prove that for the criss-cross grid

$$\tilde{f}_{i,j} \not\to f(x_i, y_j), \quad \tilde{f}_{i+\frac{1}{2},j+\frac{1}{2}} \not\to f(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}).$$

Namely, the corresponding finite element scheme is not consistent with the finite difference scheme in classic case.

2. Let V_h be a finite element space on a triangulation $\mathcal{T}_h$ of Ω. Define the norms on V_h as

$$\|w_h\|^p_{W^{2,p}_h(\Omega)} := \sum_{T\in\mathcal{T}_h} \|w_h\|^p_{W^{2,p}(T)}, \qquad \|w_h\|_{L^p_h(\Omega)} := \sup_{v_h\in V_h} \frac{(w_h, v_h)_\Omega}{\|v_h\|_{L^{p'}(\Omega)}},$$

where $1 < p < \infty$, and $1/p + 1/p' = 1$. Assume that a linear operator $\mathcal{L}_h : V_h \to V_h$ satisfies the following condition:

$$\|v_h\|_{L^{p'}(\Omega)} \lesssim \sup_{w_h\in V_h} \frac{(\mathcal{L}_h w_h, v_h)_\Omega}{\|w_h\|_{W^{2,p}_h(\Omega)}} \qquad \forall v_h \in V_h, h < h^*. \tag{2}$$

Show that for any $h < h^*$, the following stability holds:

$$\|w_h\|_{L^p(\Omega)} \lesssim \|\mathcal{L}_h w_h\|_{L^p_h(\Omega)} \qquad \forall w_h \in V_h.$$