有限元方法II，上机作业1

交作业时间：2020/01/13 晚上12:00

要求:

- 用TeX写上机报告(中英文均可)，包含必要的数值结果讨论，页数上限12.
- 本次上机作业中，要求使用软件包FEniCS, Python 或 C++均可.
- Problem 2 和 Problem 2'任选其一.
- 截止时间前将程序和上机报告的源码发送至 snwu@math.pku.edu.cn

In two-dimensional case, the de Rham complex becomes

$$\mathbb{R} \longrightarrow C^\infty(\Omega) \xrightarrow{\text{grad}} (C^\infty(\Omega))^2 \xrightarrow{\text{rot}} C^\infty(\Omega) \longrightarrow 0,$$

or

$$\mathbb{R} \longrightarrow C^\infty(\Omega) \xrightarrow{\text{curl}} (C^\infty(\Omega))^2 \xrightarrow{\text{div}} C^\infty(\Omega) \longrightarrow 0,$$

depening on which of the two identification we choose for $\Lambda^1(\mathbb{R}^2)$. Here,

$$\text{curl} v = \begin{pmatrix} \frac{\partial v}{\partial x_2} \\ -\frac{\partial v}{\partial x_1} \end{pmatrix} \qquad \text{rot} \underset{\sim}{v} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}.$$

It is straightforward to see that $\text{rot}\underset{\sim}{v} = \text{div}(\underset{\sim}{v}^\perp)$, and $(\text{grad} v)^\perp = \text{curl} v$, where $^\perp$ means rotating 90° counterclockwise. Hence, in two-dimensional case, the $H(\text{curl})$ conforming finite element is exactly the H^1 conforming finite element, and the $H(\text{rot})$ conforming finite element is the rotation of $H(\text{div})$ conforming finite element.

Problem 1

Given a computational domain $\Omega \subset \mathbb{R}^2$, consider the vector Laplace equation

$$\text{curl}\,\text{rot}\underset{\sim}{u} - \text{grad}\,\text{div}\underset{\sim}{u} = \underset{\sim}{f} \quad \text{in } \Omega, \tag{1}$$

together with the boundary condition

$$\text{div}\underset{\sim}{u} = 0, \quad \underset{\sim}{u} \times \underset{\sim}{n} = 0 \quad \text{on } \partial\Omega. \tag{2}$$

or

$$\underset{\sim}{u} \cdot \underset{\sim}{n} = 0, \quad \mathrm{rot}\underset{\sim}{u} = 0 \quad \text{on } \partial\Omega. \tag{3}$$

Here, $\underset{\sim}{u} \times \underset{\sim}{n} = u_1 n_2 - u_2 n_1$ is the tangential component of $\underset{\sim}{u}$.

1. Write down the mixed form for (1) with boundary condition (2).
2. Write down the mixed form for (1) with boundary condition (3).
3. Let $\Omega = (-1,1)^2$, choose exact solution $\underset{\sim}{u}$ to satisfy (2) or (3), and data $\underset{\sim}{f}$ to satisfy (1).
4. Using $\mathcal{P}_r\Lambda^0$-$\mathcal{P}_r^-\Lambda^1$ (Lagrange+RT) and $\mathcal{P}_r\Lambda^0$-$\mathcal{P}_{r-1}\Lambda^1$ (Lagrange+BDM) to solve (1) with boundary condition (2) or (3). Report the convergence rates of $\underset{\sim}{u}$ and σ in both energy norm and L^2 norm.
5. (Optional) Choose the L-shaped domain $\Omega = (-1,1)^2 \setminus [-1,0]^2$ and $\underset{\sim}{f} = (-1,0)$. Plot the vector field $\underset{\sim}{u}$.

Problem 2

Consider (1) with Dirichlet boundary condition $\underset{\sim}{u} = \underset{\sim}{0}$, namely,

$$\underset{\sim}{u} \cdot \underset{\sim}{n} = 0, \quad \underset{\sim}{u} \times \underset{\sim}{n} = 0 \quad \text{on } \partial\Omega. \tag{4}$$

Write down the mixed form for (1) with boundary condition (4). Using $\mathcal{P}_r\Lambda^0$-$\mathcal{P}_r^-\Lambda^1$ (Lagrange+RT) and $\mathcal{P}_r\Lambda^0$-$\mathcal{P}_{r-1}\Lambda^1$ (Lagrange+BDM), report the convergence rates of $\underset{\sim}{u}$ and σ in both energy norm and L^2 norm. Try your best to analyse your numerical results.

Problem 2'

Consider the vector Laplace with perturbation

$$\mathrm{curl}\,\mathrm{rot}\underset{\sim}{u} - \mathrm{grad}\,\mathrm{div}\underset{\sim}{u} + \boldsymbol{A}(x)\underset{\sim}{u} = \underset{\sim}{f} \quad \text{in } \Omega, \tag{5}$$

with boundary condition (2) (or (3)). Here, $\boldsymbol{A}(x)$ is a 2×2 matrix. Choose different $\underset{\sim}{A}(x)$ (constant matrix or variable matrix), report the convergence rates. You may try to analyse your numerical results OR try to do the following numerical experiments:

$$\mathrm{curl}(\mathrm{rot}\underset{\sim}{u} - \underset{\sim}{\beta} \times \underset{\sim}{u}) - \mathrm{grad}\,\mathrm{div}\underset{\sim}{u} = \underset{\sim}{f} \quad \text{in } \Omega, \tag{6}$$

with the boundary condition

$$\underset{\sim}{u} \cdot \underset{\sim}{n} = 0, \quad \mathrm{rot}\underset{\sim}{u} - \underset{\sim}{\beta} \times \underset{\sim}{u} = 0 \quad \text{on } \partial\Omega, \tag{7}$$

and report the convergence rates. Here, $\underset{\sim}{\beta}(x)$ is a given vector field.