有限元方法II，上机作业2

交作业时间：2019/12/19

要求:

- 用TeX写上机报告(中英文均可), 包含必要的数值结果讨论, 页数上限12.
- 本次上机作业中, 要求使用软件包FEniCS, Python 或 C++均可.
- 在课上提交上机报告打印版, 截止时间前将程序和上机报告的源码发送至 snwu@math.pku.edu.cn

Consider the Stokes equation

$$\begin{cases} -\mu\Delta \underset{\sim}{u} + \nabla p = \underset{\sim}{f} & \text{in } \Omega \subset \mathbb{R}^2, \\ -\nabla \cdot \underset{\sim}{u} = 0 & \text{in } \Omega, \\ \underset{\sim}{u} = \underset{\sim}{g}_D & \text{on } \Gamma_D, \\ 2\mu\varepsilon(\underset{\sim}{u})\underset{\sim}{n} - p\underset{\sim}{n} = \underset{\sim}{g}_N & \text{on } \Gamma_N = \partial\Omega \setminus \Gamma_D. \end{cases} \tag{1}$$

Problem 1. Choose $\mu = 1$, any domain $\Omega \subset \mathbb{R}^2$, exact solution $\underset{\sim}{u}$ (div-free) and p, Γ_D (Γ_D is not supposed to be $\partial\Omega$ in this problem), then $\underset{\sim}{f}$, $\underset{\sim}{g}_D$ and $\underset{\sim}{g}_N$ can be obtained. Using the following pairs

- MINI element
- $\underset{\sim}{\mathcal{P}}_2$-$\mathcal{P}_0^{-1}$
- Taylor-Hood element $\underset{\sim}{\mathcal{P}}_k$-$\mathcal{P}_{k-1}$ ($k \geq 2$)
- $\underset{\sim}{\mathcal{P}}_4$-$\mathcal{P}_3^{-1}$

Report the velocity errors in H^1, L^2, and pressure errors in L^2. Try $\underset{\sim}{\mathcal{P}}_1$-$\mathcal{P}_1$ pair and report your finding.

Problem 2. Consider the following case: $\Omega = (0,1)^2$, $\Gamma_D = \partial\Omega$. The exact solution is choosen as $\underset{\sim}{u} = \underset{\sim}{0}$, $p = y^3 - y^2/2 + y - 7/12$, hence $\underset{\sim}{f} = (0, 3y^2 - y + 1)^T$. Choose $\mu = 1, 10^{-2}, 10^{-4}, 10^{-6}$, report the velocity errors $\|\nabla(\underset{\sim}{u} - \underset{\sim}{u}_h)\|_{L^2}$ by the stable pairs in Problem 1. Discuss (and try to analyse) the results.

Problem 3. Consider the following case: $\Omega = (0,1)^2$, $\Gamma_D = \partial\Omega$. The exact solution is choosen as

$$\underset{\sim}{u} = \begin{pmatrix} x^2(1-x)^2 y(1-y)(1-2y) \\ -x(1-x)(1-2x)y^2(1-y)^2 \end{pmatrix},$$
$$p = 10\left((x-1/2)^3 y^2 + (1-x)^3(y-1/2)^3\right).$$

Using the "grad-div stabilization" in the variation form, namely

$$a_h(\underset{\sim}{u}_h, \underset{\sim}{v}_h) = (2\mu\varepsilon(\underset{\sim}{u}_h), \varepsilon(\underset{\sim}{v}_h)) + \gamma(\nabla\cdot\underset{\sim}{u}_h, \nabla\cdot\underset{\sim}{v}_h)$$

Using the Taylor-Hood pair $\underset{\sim}{\mathcal{P}}_2$-$\mathcal{P}_1$. For $\mu = 1, 10^{-2}, 10^{-4}$, report the $\|\nabla(\underset{\sim}{u} - \underset{\sim}{u}_h)\|_{L^2}$ and $\|\nabla\cdot\underset{\sim}{u}_h\|_{L^2}$ for various γ. Discuss the results.