有限元方法II，作业6

交作业时间：2019/12/26

1. Let $\Omega \subset \mathbb{R}^3$ be the half space $x_3 < 0$ and Γ be the space $x_3 = 0$. Given a vector $\underset{\sim}{\chi} = (\chi_1, \chi_2, \chi_3)^T \in H(\mathrm{curl}, \Omega)$, show that $\mathrm{div}_\Gamma \mathrm{Tr}\underset{\sim}{\chi} = \mathrm{curl}\underset{\sim}{\chi} \cdot \underset{\sim}{n}|_\Gamma$, where $\mathrm{Tr}\chi = \underset{\sim}{n} \times \underset{\sim}{\chi}$.

2. Show that for any $\underset{\sim}{p}_k \in \underset{\sim}{\mathcal{P}}_k$, there exists a decomposition

$$\underset{\sim}{p}_k = \underset{\sim}{w}_{k-1} + \nabla\theta,$$

where $\underset{\sim}{w}_{k-1} \in N_{k-1} := \underset{\sim}{\mathcal{P}}_{k-1} + \underset{\sim}{x} \times \underset{\sim}{\mathcal{P}}_{k-1}$.

3. Given any $\epsilon > 0$, let $X = H(\mathrm{curl}) \cap \underset{\sim}{H}^{1/2+\epsilon}$. Given a Lipschitz domain Ω, show that there exists $\delta(\epsilon, \Omega) > 0$ so that $\mathrm{curl}X(\Omega) \subset \underset{\sim}{L}^{2+\delta(\epsilon,\Omega)}(\Omega)$. Let $W = H(\mathrm{div}) \cap \underset{\sim}{L}^{2+\delta(\epsilon)}$. Show the following commutative diagram

$$\begin{array}{ccc} X(K) & \xrightarrow{\mathrm{curl}} & W(K) \\ \Pi_k^N \big\downarrow & & \big\downarrow \Pi_k^{RT} \\ N_k(K) & \xrightarrow{\mathrm{curl}} & RT_k(K) \end{array} \qquad \begin{array}{ccc} X(K) & \xrightarrow{\mathrm{curl}} & W(K) \\ \Pi_{k+1}^{NC} \big\downarrow & & \big\downarrow \Pi_k^{RT} \\ NC_{k+1}(K) & \xrightarrow{\mathrm{curl}} & RT_k(K) \end{array}$$

Find the similar version for $BDM_k(K)$ (no need to show the proof).

4. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. Show that the pullback $L^* : \mathrm{Alt}^2\mathbb{R}^3 \to \mathrm{Alt}^2\mathbb{R}^3$ has the matrix form $(\det L)L^{-1} : \mathbb{R}^3 \to \mathbb{R}^3$ by the correspondence between $\mathrm{Alt}^2\mathbb{R}^3$ and $\mathbb{R}^3$.

5. Show the Leibniz rule

$$\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^k \omega \wedge \mathrm{d}\eta, \qquad \omega \in \Lambda^k(\Omega), \eta \in \Omega^j(\Omega).$$

6. For any increasing sequence $\sigma : \{1, \cdots, k\} \to \{1, \cdots, n\}$, show that

$$\mathrm{d}(a_\sigma \mathrm{d}x_{\sigma(1)} \wedge \cdots \mathrm{d}x_{\sigma(k)}) = \sum_{i=1}^n \frac{\partial a_\sigma}{\partial x_i} \mathrm{d}x_i \wedge \mathrm{d}x_{\sigma(1)} \wedge \cdots \mathrm{d}x_{\sigma(k)}.$$

7. Using the Poincaré inequality, show that

$$\|\omega\| \lesssim \|\delta\omega\|, \quad \forall \omega \in \mathfrak{Z}^{*k,\perp}.$$