有限元方法II，作业5

交作业时间：2019/12/11

1. Let Ω be a connected domain with a Lipschitz boundary. Given a space $H_0^1(\Omega) \subset V \subset H_1(\Omega)$, if $V \not\subset \{\utilde{u} \mid \int_{\partial\Omega} \utilde{u} \cdot \utilde{n} \mathrm{d}s = 0\}$, then the following inf-sup condition holds
$$\inf_{q \in L^2(\Omega)} \sup_{\utilde{v} \in V} \frac{(\mathrm{div}\utilde{v}, q)}{\|\utilde{v}\|_{\utilde{H}^1(\Omega)} \|q\|_{L^2(\Omega)}} = \beta > 0.$$

2. Let Ω be a connected domain with a Lipschitz boundary. Assume that $\Gamma_D \subset \partial\Omega$ satisfies $\mathrm{meas}(\Gamma_D) \neq 0$. Show that
$$\|\utilde{v}\|_{\utilde{H}^1(\Omega)} \lesssim \|\boldsymbol{\varepsilon}(\utilde{v})\|, \quad \forall \utilde{v} \in \utilde{H}_D^1(\Omega),$$
where $\utilde{H}_D^1(\Omega) := \{\utilde{v} \in \utilde{H}^1(\Omega) : \ \utilde{v} = 0 \text{ on } \Gamma_D\}$.

3. Let $\mathcal{T}_h$ is a shape-regular triangulation of $\Omega \subset \mathbb{R}^2$. For any edge e connected by V_i and V_j, define
$$b_e := 6\phi_i(x)\phi_j(x)/|e|,$$
where $\phi_i(x)$ is the piecewise linear basis function associated with V_i. Let
$$\Pi_h \utilde{v} := \sum_e \left(\int_e \utilde{v} \mathrm{d}s \right) b_e.$$
Show that
$$\|\Pi_h \utilde{v}\|_{L^2}^2 \lesssim \|\utilde{v}\|_{L^2}^2 + h^2 |\utilde{v}|_{H^1}^2.$$

4. For the Stokes problem, assume the modified inf-sup condition
$$\inf_{q_h \in Q_h} \sup_{\utilde{v}_h \in V_h} \frac{(\mathrm{div}\utilde{v}_h, q_h)}{\|\utilde{v}_h\|_V \|q_h - \bar{q}_h\|_Q} = k_0 > 0,$$
where k_0 is independent of h and $\bar{q}_h$ is the L^2 projection of q_h. Assume further that V_h is such that, for any $q_h \in \mathcal{P}_0^{-1}$,
$$\sup_{\utilde{v}_h \in V_h} \frac{(\mathrm{div}\utilde{v}_h, q_h)}{\|\utilde{v}_h\|_V} \geq \gamma_0 \|q_h\|_Q,$$
with γ_0 independent of h. Then, $V_h \times Q_h$ is inf-sup stable.

5. Let $\hat{K} \subset \mathbb{R}^n$, F be a smooth mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$, and $K = F(\hat{K})$. Assume that F is globally invertible on K and its Jacobian DF is invertible. For any $\underset{\sim}{\hat{q}} \in (C^\infty(\hat{K}))^n$, define

$$\mathcal{G}(\underset{\sim}{\hat{q}})(x) := \frac{1}{J(\hat{x})} DF(\hat{x}) \underset{\sim}{\hat{q}}(\hat{x}), \quad \hat{x} = F^{-1}(x),$$

where $J(x) = |\det DF(\hat{x})|$. Show that

$$\mathrm{div} \underset{\sim}{q} = \frac{1}{J} \widehat{\mathrm{div}} \underset{\sim}{\hat{q}}.$$

Here, $\widehat{\mathrm{div}}$ means the derivatives on $\hat{x}$.

6. Let $v = \mathcal{F}(\hat{v}) := \hat{v}(F^{-1}(x))$, and $\underset{\sim}{q} = \mathcal{G}(\underset{\sim}{\hat{q}})$, where $F(\hat{x}) = B\hat{x} + b_0$ is an affine mapping. Show that

$$\begin{aligned} \int_K \underset{\sim}{q} \cdot \mathrm{grad} v \, \mathrm{d}x &= \int_{\hat{K}} \underset{\sim}{\hat{q}} \cdot \hat{\mathrm{grad}} \hat{v} \, \mathrm{d}\hat{x}, \\ \int_K v \, \mathrm{div} \underset{\sim}{q} \, \mathrm{d}x &= \int_{\hat{K}} \hat{v} \, \hat{\mathrm{div}} \underset{\sim}{\hat{q}} \, \mathrm{d}\hat{x}, \\ \int_{\partial K} \underset{\sim}{q} \cdot \underset{\sim}{n} v \, \mathrm{d}s &= \int_{\partial \hat{K}} \underset{\sim}{\hat{q}} \cdot \underset{\sim}{\hat{n}} \hat{v} \, \mathrm{d}\hat{s}. \end{aligned}$$

7. Define

$$\mathbb{H}_k(K) := \{ \underset{\sim}{q} \in \underset{\sim}{\mathcal{P}}_k(K) \mid \mathrm{div} \underset{\sim}{q} = 0 \quad \text{and} \quad \underset{\sim}{q} \cdot \underset{\sim}{n}|_{\partial K} = 0 \}.$$

Show that in 2D, $\mathbb{H}_k(K) = \mathrm{curl}(b_K \mathcal{P}_{k-2}(K))$.

8. Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain. For any $\underset{\sim}{v} \in (L^2(\Omega))^2$, show that there exists $\psi \in H_0^1(\Omega)$ and $\phi \in H^1(\Omega)$, such that

$$\underset{\sim}{v} = \nabla \psi + \mathrm{curl} \phi,$$

and

$$\|\nabla \phi\|_{L^2} + \|\nabla \psi\|_{L^2} \lesssim \|\underset{\sim}{v}\|_{L^2}.$$

(Hint: $R(\mathrm{curl}) = N(\mathrm{div})$ on simply connected domain.)