有限元方法II，作业4

交作业时间：2019/11/27

1. Let H be a Hilbert space with a norm $\|\cdot\|_H$ and inner product $(\cdot,\cdot)_H$. Let $P : H \to H$ be an idempotent, such that $0 \neq P^2 = P \neq I$. Then, the following indentity holds

$$\|P\|_{\mathcal{L}(H,H)} = \|I - P\|_{\mathcal{L}(H,H)}.$$

2. Assume that $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite and $B \in \mathbb{R}^{m\times n}$ is of full-rank m ($m \leq n$). Let $S = BA^{-1}B^T$. Show that the "preconditioned" matrix

$$\begin{pmatrix} A^{-1} & 0 \\ 0 & S^{-1} \end{pmatrix} \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}$$

only has three distinctive eigenvalues: $1, (1 \pm \sqrt{5})/2$. Determine the multiplicity of these eigenvalues.

3. Let V, Q be Banach spaces, $B : V \to Q'$ be a bound linear operator, $Z := N(B)$. For any $S \subset V$, define

$$S^\circ := \{f \in V' \mid \langle f, v\rangle = 0, \ \forall v \in S\}.$$

For any $F \subset V'$, define

$${}^\circ F := \{v \in V \mid \langle f, v\rangle = 0, \ \forall f \in F\}.$$

- Show that S° and ${}^\circ F$ are closed.
- Show that ${}^\circ(S^\circ) = S$ if and only if S is closed in V; And $({}^\circ F)^\circ = F$ if and only if F is closed in V'.
- Show that ${}^\circ R(B') = Z$.
- Show that $R(B') = Z^\circ$ if and only if $R(B')$ is closed in V'.

4. Let V, Q be Banach spaces, $b(\cdot,\cdot) : V \times Q \to \mathbb{R}$ be a bilinear form. Show that the inf-sup condition

$$\inf_{q\in Q} \sup_{v\in V} \frac{b(v,q)}{\|v\|\|q\|} = \beta > 0$$

is equivalent to the following statement: $\forall q \in Q$, $\exists v \in V$ such that

$$b(v,q) = \|q\|^2 \quad \text{and} \quad \|v\| \leq C\|q\|.$$

5. For any $p \in L^2(\Omega)$, show that

$$\|p\|_{H^{-1}} + \sum_{i=1}^{d} \|\frac{\partial p}{\partial x_i}\|_{H^{-1}} \lesssim \|p\|_{L^2}.$$

6. Consider the Stokes problem with homogeneous Dirichlet boundary condition:

$$\begin{aligned} -\Delta \underset{\sim}{u} + \nabla p &= \underset{\sim}{f} \quad \text{in } \Omega, \\ -\text{div}\underset{\sim}{u} &= 0 \quad \text{in } \Omega, \\ \underset{\sim}{u}|_{\partial\Omega} &= \underset{\sim}{0}. \end{aligned}$$

Let $V = \underset{\sim}{H}_0^1(\Omega)$ and $Q = L_0^2(\Omega)$. Given a stable Stokes pair $V_h \times Q_h \subset V \times Q$, we can obtain the following energy energy estimate

$$\|\underset{\sim}{u} - \underset{\sim}{u}_h\|_{H^1} + \|p - p_h\|_{L^2} \lesssim \inf_{\underset{\sim}{v}_h \in V_h} \|\underset{\sim}{u} - \underset{\sim}{v}_h\|_{H^1} + \inf_{q_h \in Q_h} \|p - q_h\|_{L^2}.$$

Assume further the approximation property of $V_h \times Q_h$:

$$\begin{aligned} \inf_{\underset{\sim}{v}_h \in V_h} \|\underset{\sim}{z} - \underset{\sim}{v}_h\|_{H^1} &\lesssim h\|\underset{\sim}{z}\|_{H^2} \quad \forall \underset{\sim}{z} \in \underset{\sim}{H}^2(\Omega), \\ \inf_{q_h \in Q_h} \|r - q_h\|_{L^2} &\lesssim h\|r\|_{H^1}, \quad \forall r \in H^1(\Omega). \end{aligned}$$

<u>Duality argument</u>: Find appropriate regularity assumption of the dual problem:

$$\begin{aligned} -\Delta \underset{\sim}{z} + \nabla r &= \underset{\sim}{\theta} \quad \text{in } \Omega, \\ \text{div}\underset{\sim}{z} &= 0 \quad \text{in } \Omega, \\ \underset{\sim}{z}|_{\partial\Omega} &= \underset{\sim}{0}. \end{aligned}$$

so that one can obtain the L^2 estimate of $\underset{\sim}{u}$:

$$\|\underset{\sim}{u} - \underset{\sim}{u}_h\|_{L^2} \lesssim h \left(\inf_{\underset{\sim}{v}_h \in V_h} \|\underset{\sim}{u} - \underset{\sim}{v}_h\|_{H^1} + \inf_{q_h \in Q_h} \|p - q_h\|_{L^2} \right).$$