差分方法II，上机作业

截止时间：2024/06/14晚上12点

要求:

- 用TeX写上机报告(中英文均可), 包含必要的数值结果讨论, 页数上限20.
- 程序语言不限, 但需要说明如何编译运行程序 (包含README文件或者在上机报告中说明).
- 截止时间前将程序源码和上机报告发送至snwu@math.pku.edu.cn

Solve the following Monge-Ampère equation in 2D using the two-scale (or wide stencil FD) method:

$$\begin{cases} \det D^2 u = f & \text{in } \Omega \subset \mathbb{R}^2, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

The computational domain is the unit square, i.e., $\Omega = (0,1)^2$. Let $\boldsymbol{x} = (x,y)^T$, $\boldsymbol{x}_0 = (0.5, 0.5)^T$. The numerical experiments are suggested to be conducted with three different examples:

- Smooth and radial example:

$$u(\boldsymbol{x}) = \exp(|\boldsymbol{x}|^2/2), \quad f(\boldsymbol{x}) = (1+|\boldsymbol{x}|^2)\exp(|\boldsymbol{x}|^2).$$

- C^1 example:

$$u(\boldsymbol{x}) = \frac{1}{2}\left((|\boldsymbol{x}-\boldsymbol{x}_0| - 0.2)^+\right)^2, \quad f(\boldsymbol{x}) = \left(1 - \frac{0.2}{|\boldsymbol{x}-\boldsymbol{x}_0|}\right)^+.$$

- Twice differentiable in the interior domain, but has unbounded gradient near the boundary point $(1,1)$:

$$u(\boldsymbol{x}) = -\sqrt{2-|\boldsymbol{x}|^2}, \quad f(\boldsymbol{x}) = 2(2-|\boldsymbol{x}|^2)^{-2}.$$

The boundary data $g(\boldsymbol{x})$ can be obtained from the solution.

1. In this lab, the Perron's iteration and Newton's method [1] are necessary for solving the nonlinear system resulting from discretization. Selecting suitable parameters $\delta = h^\alpha$, $\theta = h^\beta$ for some α, β (or approximate parameters in wide stencil FD), then report and discuss the errors in L^∞ for different parameter choices.

2. You can try one or more of the following (even multiple) to earn extra credit:

   - Can higher-order difference schemes (combined with the stability of monotone schemes) be used to achieve better accuracy? For example, the filtered scheme proposed in [2].
   - For solving nonlinear problems, can the idea of multigrid be utilized to improve computational efficiency?
   - If the boundary conditions are not Dirichlet but transport boundary conditions (see [3]), can a corresponding two-scale method be designed?
   - Any other extensions related to this computer assignment that you can think of.

References

[1] Ricardo H. Nochetto, Dimitrios Ntogkas, and Wujun Zhang. *Two-scale method for the Monge-Ampère equation: convergence to the viscosity solution*, **Mathematics of Computation**, 88(316), 637-664, 2019.

[2] Ricardo H. Nochetto, and Dimitrios Ntogkas. *Convergent two-scale filtered scheme for the Monge-Ampère equation*, **SIAM Journal on Scientific Computing**, 41(2), B295-B319, 2019.

[3] Froese, Brittany D. *A numerical method for the elliptic Monge–Ampère equation with transport boundary conditions*, **SIAM Journal on Scientific Computing** 34(3), A1432-A1459, 2012.