差分方法II，作业7

交作业时间方式：2024/06/05之前，扫描发送至snwu@math.pku.edu.cn

1. For ODE $y' = f(t, y)$, consider the linear multi-step method

   $$a_k y_{n+k} + a_{k-1} y_{n+k-1} + \ldots + a_0 y_n = b_k f_{n+k} + b_{k-1} f_{n+k-1} + \ldots + b_0 f_n.$$

   Show that the multistep is of order p

   $\Longleftrightarrow$ (i) $\sum_{i=0}^{k} a_i = 0$ and $\sum_{i=0}^{k} a_i i^q = q \sum_{i=0}^{k} b_i i^{q-1}, q = 1, \ldots, p;$

   $\Longleftrightarrow$ (ii) $\rho(\mathrm{e}^\tau) - \tau\sigma(\mathrm{e}^\tau) = \mathcal{O}(\tau^{p+1})$ as $\tau \to 0;$

   $\Longleftrightarrow$ (ii) $\frac{\rho(\zeta)}{\log(\zeta)} - \sigma(\zeta) = \mathcal{O}((\zeta - 1)^p)$ as $\zeta \to 1.$

   Here, $\rho(\zeta)$ and $\sigma(\zeta)$ are the generating functions of $\{a_i\}_{i=0}^k$ and $\{b_i\}_{i=0}^k$, respectively.

2. Assume $u(t)$ is continuous on $[0, T]$ and $u(0) = 0$. Show that

   $$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} \partial_\xi^\alpha u(\xi) \mathrm{d}\xi.$$

   Notice that the Caputo and Riemann-Liouville fractional derivatives coincide due to $u(0) = 0$.

3. Consider the two-parameter Milttag-Leffler function

   $$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)} \quad z \in \mathbb{C}.$$

   For $\lambda > 0$, $\alpha > 0$, and $t > 0$, show that

   $$\begin{aligned} {}^C\partial_t^\alpha E_{\alpha,1}(-\lambda t^\alpha) &= -\lambda E_{\alpha,1}(-\lambda t^\alpha), \\ E_{\alpha,1}(-\lambda t^\alpha) &= 1 - \lambda t^\alpha E_{\alpha,1+\alpha}(-\lambda t^\alpha). \end{aligned}$$

4. Assume that $K(z)$ and $\delta_\tau(\zeta)$ satisfy the conditions for parabolic type in CQ. Verify that in the standard CQ for the incompatible part, the $\frac{t^\ell}{\ell!}$ term, yields the following result:

   $$K(z) z^{-\ell-1} + K(\delta_\tau(e^{-\tau z})) \tau^{\ell+1} \eta_\ell(e^{-\tau z}) \quad \text{is not } \mathcal{O}(\tau^k),$$

   where

   $$\eta_\ell(\zeta) := \frac{1}{\ell!} \left[\left(\zeta \frac{\mathrm{d}}{\mathrm{d}\zeta}\right)^\ell \frac{1}{1 - \zeta} \right].$$

5. Given $\theta \in (\frac{\pi}{2}, \pi)$, let

$$\Sigma_\theta(\omega) := w + \{z \in \mathbb{C} : |\mathrm{arg} z| < \theta\},$$
$$\Gamma_\theta(\omega) := \partial\Sigma_\theta(\omega), \quad \Gamma_\theta^\tau(\omega) := \{z \in \Gamma_\theta : |\mathrm{Im}(z)| \le \frac{\pi}{\tau}\}.$$

Show the following estimates (for some $\omega > 0$):

$$\int_{\Gamma_\theta(\omega)\setminus\Gamma_\theta^\tau(\omega)} |z|^{\beta-\ell-1} \mathrm{e}^{t_n \mathrm{Re}(z)} |\mathrm{d}z| \le C\tau^k t_n^{\ell-\beta-k},$$
$$\int_{\Gamma_\theta^\tau(\omega)} |z|^{\beta-\ell+k-1} \mathrm{e}^{t_n \mathrm{Re}(z)} |\mathrm{d}z| \le C t_n^{\ell-k-\beta}.$$