2024春, 差分方法II，作业6

交作业时间：2024/05/24

1. Assume that Ω is triangulated as $\mathcal{T}_h$. Let $R > 0$, so that Ω satisfies
$$\Omega \subset \{\boldsymbol{x} \mid x_1^2 + \cdots + x_{d-1}^2 + (x_d - R)^2 < R^2\},$$
and the two are tangent at the origin. For the point $\boldsymbol{x}_0 = (0, \cdots, 0, z)$ on the d-axis, where $0 < z < \delta$, consider the following quadratic function:
$$p(x) = \frac{E^{1/d}}{2}(x_1^2 + \cdots + x_{d-1}^2 + (x_d - R)^2 - R^2).$$
Show that the Lagrange interpolation of $p(x)$ under $\mathcal{T}_h$, denoted as $p_h := I_h p$, satisfies
$$\begin{aligned} T_\varepsilon[p_h](x_i) &\geq E \quad \forall x_i \in \mathcal{N}_h^0, \\ p_h\,|_{\partial\Omega_h} &\leq 0, \\ |p_h(z)| &\leq C E^{1/d} \delta. \end{aligned}$$

2. Let v_h be a piecewise linear function on the mesh $\mathcal{T}_h$. For any face $F = T^+ \cap T^-$ with $T^\pm \in \mathcal{T}_h$, define the jump
$$[\![v_h]\!]|_F := -n_F^+ \cdot \nabla v_h|_{T^+} - n_F^- \cdot \nabla v_h|_{T^-}.$$
Show that v_h is convex if and only if
$$[\![v_h]\!]|_F \geq 0 \qquad \text{for all faces } F.$$

3. Let $p(x) = \frac{1}{2}(x^2 + y^2)$. A mesh $\mathcal{T}_h$ is called *Delaunay* if the sum of the angles opposite to any edge is less than or equal to π. Show that $I_h p = \Gamma(I_h p)$ if and only if $\mathcal{T}_h$ is Delaunay, where I_h is the interpolant to the continuous piecewise linear space.

4. Let $\mathbf{S}_+ := \{A \in \mathbf{S}, A \geq 0\}$ and $\mathbf{S}_1 = \{B \in \mathbf{S}_+, \operatorname{tr}(B) = 1\}$, where $\mathbf{S}$ denotes the set of $d \times d$ symmetric real matrices. Define the Bellman operator
$$H(A, f) := \sup_{B \in \mathbf{S}_1} \left(-B : A + f \sqrt[d]{\det B}\right) \quad \forall A \in \mathbf{S}, f \in [0, \infty),$$
and the Monge-Ampère operator
$$M(A, f) := \left(\frac{f}{d}\right)^d - \det A \quad \forall A \in \mathbf{S}, f \in [0, \infty).$$

(a) Let $f \in [0, \infty)$ and $A \in \mathbf{S}$. Show that $H(A, f) = 0$ holds if and only if $M(A, f) = 0$ and $A \in \mathbf{S}_+$.

(b) Let $A \in \mathbf{S}_+$, $f \in [0, \infty)$ and let λ be the smallest eigenvalue of A. Show that the function

$$\Phi_{A,f} : [-f, \infty) \to [-\lambda, \infty), \delta \mapsto H(A, f + \delta)$$

is continuous, strictly monotonically increasing and bijective.